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On the mean value of $SSMP(n)$ and $SIMP(n)$[1]

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Abstract The main purpose of this paper it to studied the mean value properties of the Smarandache Superior m-th power part sequence $SSMP(n)$ and the Smarandache Inferior m-th power part sequence $SIMP(n)$, and give several interesting asymptotic formula for them.

Keywords Smarandache Superior m-th power part sequence, Smarandache Inferior m-th power part sequences, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the Smarandache Superior m-th power part sequence $SSMP(n)$ is defined as the smallest m-th power greater than or equal to n. The Smarandache Inferior m-th power part sequence $SIMP(n)$ is defined as the largest m-th power less than or equal to n. For example, if $m = 2$, then the first few terms of $SIMP(n)$ are: 0, 1, 1, 1, 4, 4, 4, 4, 4, 9, 9, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 16, 16, 25, $\cdots$. The first few terms of $SSMP(n)$ are: 1, 4, 4, 4, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 25, $\cdots$. If $m = 3$, then The first few terms of $SSMP(n)$ are: 1, 8, 8, 8, 8, 8, 8, 8, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 64, $\cdots$. The first few terms of $SIMP(n)$ are: 0, 1, 1, 1, 1, 1, 1, 1, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 27, $\cdots$. Now we let

$$\begin{aligned} S_n &= (SSMP(1) + SSMP(2) + \cdots + SSMP(n))/n; \\ I_n &= (SIMP(1) + SIMP(2) + \cdots + SIMP(n))/n; \\ K_n &= \sqrt[n]{SSMP(1) + SSMP(2) + \cdots + SSMP(n)}; \\ I_n &= \sqrt[n]{SIMP(1) + SIMP(2) + \cdots + SIMP(n)}. \end{aligned}$$

In reference [2], Dr. K.Kashihara asked us to study the properties of these sequences. Gou Su [3] studied these problem, and proved the following conclusion:

For any real number $x > 2$ and integer $m = 2$, we have the asymptotic formula

$$\sum_{n\leqslant x} SSSP\left(n\right) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right), \quad \sum_{n\leqslant x} SISP\left(n\right) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right),$$

and

$$\tfrac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{2}}\right), \quad \lim_{n\to\infty} \tfrac{S_n}{I_n} = 1.$$

[1]This work is supported by the Shaanxi Provincial Education Department Foundation 08JK433.

In this paper, we shall use the elementary method to give a general conclusion. That is, we shall prove the following:

Theorem 1. Let $m \geq 2$ be an integer, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right),$$

and

$$\sum_{n \leq x} SIMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right).$$

Theorem 2. For any fixed positive integer $m \geq 2$ and any positive integer n, we have the asymptotic formula

$$S_n - I_n = \frac{m(m-1)}{2m-1} n^{1-\frac{1}{m}} + O\left(n^{1-\frac{2}{m}}\right).$$

Corollary 1. For any positive integer n, we have the asymptotic formula

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{m}}\right),$$

and the limit $\lim\limits_{n\to\infty} \frac{S_n}{I_n} = 1$.

Corollary 2. For any positive integer n, we have the asymptotic formula

$$\frac{K_n}{L_n} = 1 + O\left(\frac{1}{n}\right),$$

and the limit $\lim\limits_{n\to\infty} \frac{K_n}{L_n} = 1$, $\lim\limits_{n\to\infty} (K_n - L_n) = 0$.

§2. Proof of the theorems

In this section, we shall use the Euler summation formula and the elementary method to complete the proof of our Theorems. For any real number $x > 2$, it is clear that there exists one and only one positive integer M satisfying $M^m < x \leq (M+1)^m$. That is, $M = x^{\frac{1}{m}} + O(1)$. So we have

$$\begin{aligned}
\sum_{n \leq x} SSMP(n) &= \sum_{n \leq M^m} SSMP(n) + \sum_{M^m < n \leq x} SSMP(n) \\
&= \sum_{k \leq M} (k^m - (k-1)^m) k^m + ([x] - (M^m + 1))(M+1)^m \\
&= \sum_{k \leq M} (mk^{2m-1} + O(k^{2m-2})) + ([x] - M^m - 1)(M+1)^m \\
&= \frac{m \cdot M^{2m}}{2m} + O\left(M^{2m-1}\right) + ([x] - M^m - 1)(M+1)^m \\
&= \frac{M^{2m}}{2} + O\left(M^{2m-1}\right).
\end{aligned}$$

Note that $M = x^{\frac{1}{m}} + O(1)$, from the above estimate we have the asymptotic formula

$$\sum_{n\le x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2-\frac{1}{m}}\right).$$

This proves the first formula of Theorem 1.

Now we prove the second one. For any real number $x > 1$, we also have

$$\begin{aligned}\sum_{n\le x} SIMP(n) &= \sum_{n<M^m} SIMP(n) + \sum_{M^m\le n\le x} SIMP(n)\\ &= \sum_{k\leqslant M} (k^m-(k-1^m))(k-1)^m + \sum_{M^m\leqslant n\leqslant x} M^m\\ &= \sum_{k\leqslant M} (mk^{2m-1}+O(k^{2m-2})) + ([x]-M^m+1)\, M^m\\ &= \frac{M^{2m}}{2} + O\left(M^{2m-1}\right) + ([x]-M^m+1)\, M^m.\end{aligned}$$

Note that

$$([x]-M^m+1)\, M^m \leqslant M^{2m-1} \le x^{1-\frac{1}{m}}.$$

Therefore,

$$\sum_{n\le x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2-\frac{1}{m}}\right).$$

This completes the proof of Theorem 1.

To prove Theorem 2, let $x = n$, then from the method of proving Theorem 1 we have

$$\begin{aligned}S_n - I_n &= \frac{1}{n}\left(SSMP(1)+SSMP(2)+\cdots+SSMP(n)\right)\\ &\quad -\frac{1}{n}\left(SIMP(1)+SIMP(2)+\cdots+SIMP(n)\right)\\ &= \frac{1}{n}\left(\sum_{k\le M}(k^m-(k-1)^m)k^m + ([x]-(M^m+1))(M+1)^m\right)\\ &\quad -\frac{1}{n}\left(\sum_{k\leqslant M}(k^m-(k-1^m))(k-1)^m + ([x]-M^m+1)\, M^m\right)\\ &= \frac{1}{n}\sum_{k\le M} m(m-1)k^{2m-2} + O\left(\frac{1}{n}M^{2m-2}\right)\\ &= \frac{m(m-1)}{n(2m-1)}M^{2m-1} + O\left(\frac{1}{n}M^{2m-2}\right).\end{aligned}$$

Note that $M^m < n \le (M+1)^m$ or $M = n^{\frac{1}{m}} + O(1)$, from the above formula we may immediately deduce that

$$S_n - I_n = \frac{m(m-1)}{2m-1}n^{1-\frac{1}{m}} + O\left(n^{1-\frac{2}{m}}\right).$$

This completes the proof of Theorem 2.

Now we prove the Corollaries. Note that the asymptotic formula

$$I_n = \frac{1}{n}(SIMP(1) + SIMP(2) + \cdots + SIMP(n)) = \frac{1}{n}\left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right) = \frac{n}{2} + O\left(n^{1-\frac{1}{m}}\right)$$

and

$$S_n = \frac{1}{n}\left(SSMP(1) + SSMP(2) + \cdots + SSMP(n)\right) = \frac{1}{n}\left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right) = \frac{n}{2}+O\left(n^{1-\frac{1}{m}}\right).$$

From the above two formula we have

$$\frac{S_n}{I_n} = \frac{\frac{n}{2} + O\left(n^{\frac{m-1}{m}}\right)}{\frac{n}{2} + O\left(n^{\frac{m-1}{m}}\right)} = 1 + O\left(n^{-\frac{1}{m}}\right).$$

Therefore, we have the limit formula

$$\lim_{n\to\infty} \frac{S_n}{I_n} = 1.$$

Using the same method we can also deduce that

$$K_n = \sqrt[n]{SSMP(1) + SSMP(2) + \cdots + SSMP(n)} = \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right)^{\frac{1}{n}}$$

and

$$L_n = \sqrt[n]{SIMP(1) + SIMP(2) + \cdots + SIMP(n)} = \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right)^{\frac{1}{n}}$$

From these formula we may immediately deduce that

$$\frac{K_n}{L_n} = \left(\frac{\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)}{\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)}\right)^{\frac{1}{n}} = \left(1 + O\left(n^{-\frac{1}{m}}\right)\right)^{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right).$$

Therefore, we have the limit formula

$$\lim_{n\to\infty} \frac{K_n}{L_n} = 1.$$

Note that $\lim_{n\to\infty} K_n = \lim_{n\to\infty} L_n = 1$, we may immediately deduce that

$$\lim_{n\to\infty} (K_n - L_n) = 0.$$

This completes the proof of Corollary 2.

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A family of Beta-Fibonacci sequences

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Abstract This paper gives a generalization of Beta-nacci and Fibonacci sequences and the general solution obtained is given in terms of Beta-Fibonacci numbers.

Keywords Beta-nacci sequence, Fibonacci sequence.

§1. Preliminaries and introduction

The Fibonacci sequence, say $\{F_n\}_{n=0}^{\infty}$ is defined recurrently by

$$F_n = F_{n-1} + F_{n-2}, \qquad \text{for all } n \geq 2, \tag{1}$$

with initial conditions

$$F_0 = 1; \quad F_1 = 1.$$

The general term of the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

The Fibonacci sequence has been studied extensively and generalized in many ways. In [1] Peter R. J. Asveld studied the class of recurrence relations

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j \tag{2}$$

with initial conditions

$$G_0 = 1; \quad G_1 = 1.$$

The main result of [1] consists of an expression for G_n in terms of the Fibonacci numbers F_n and F_{n-1}, and in the parameters $\alpha_0, \alpha_1, ..., \alpha_k$.

The Beta-nacci sequence, say $\{B_n\}_{n=0}^{\infty}$ is defined recurrently by

$$B_n = B_{n-1} + 2B_{n-2}, \qquad \text{for all } n \geq 2, \tag{3}$$

with initial conditions

$$B_0 = 1; \quad B_1 = 1.$$

The general term of the Beta-nacci sequence is

$$B_n = \frac{2^{n+1} + (-1)^n}{3}.$$

In this paper, we give a generalized of Beta-nacci sequences and the general solution obtained is given in terms of Beta-nacci numbers. Also we consider a generalization of the Fibonacci and the Beta-nacci sequences, then we define a new recurrence, which we call the Beta-Fibonacci sequence. Further, we give a generalization of Beta-Fibonacci sequence, called the generalized BF-nacci sequence and express the n^{th} term of the generalized BF-nacci sequence in terms of the Beta-Fibonacci numbers.

§2. A generalization of Beta-nacci sequence

Definition. For any non-negative integer k and any real numbers $\alpha_0, \alpha_1, \cdots, \alpha_k$, a generalization of Beta-nacci sequence $\{S_n\}_{n=0}^{\infty}$ is defined recurrently by

$$S_n = S_{n-1} + 2S_{n-2} + \sum_{j=0}^{k} \alpha_j n^j, \qquad \text{for all } n \geq 2, \tag{4}$$

with initial conditions

$$S_0 = 1; S_1 = 1.$$

Theorem 1. The solution of (4) can be express as

$$S_n = (1 - \Lambda_k)B_n + \Psi_k B_{n-1} + \sum_{j=0}^{k} p_j(n)\alpha_j, \tag{5}$$

where

(i) Λ_k is a linear combination of $\alpha_0, \alpha_1, \cdots, \alpha_k$;

(ii) Ψ_k is a linear combination of $\alpha_1, \cdots, \alpha_k$;

(iii) for each j ($0 \leq j \leq k$), $p_j(n)$ is a polynomail of degree j.

Proof of Theorem 1. First, we solve the homogeneous recurrence relation

$$S_n = S_{n-1} + 2S_{n-2}.$$

The characteristic polynomail, $x^2 - x - 2$, has distinct roots 2 and -1, so the solution is

$$S_n^{(h)} = c_1 2^n + c_2(-1)^n,$$

where c_1 and c_2 are constants.

Next we find the particular solution of (4).

We set $S_n^{(p)} = \sum_{i=0}^{k} A_i n^i$, and attempt to determine $A_0, A_1, ..., A_k$.

Putting this expression for $S_n^{(p)}$ in (4), we obtain

$$\sum_{i=0}^{k} A_i n^i = \sum_{i=0}^{k} A_i (n-1)^i + 2\sum_{i=0}^{k} A_i (n-2)^i + \sum_{i=0}^{k} \alpha_i n^i.$$

Hence, for each $i = 0, 1, \cdots, k$, we get

$$A_i - \sum_{m=i}^{k} \beta_{im} A_m - \alpha_i = 0 \tag{6}$$

with, for $m \geq i$,

$$\beta_{im} = \binom{m}{i} (-1)^{m-i} (1 + 2^{m-i+1}).$$

From the recurrence relation (6), we can successively determine $A_k, A_{k-1}, \cdots, A_0$: the coefficient A_i is a linear combination of $\alpha_i, \alpha_{i+1}, \cdots, \alpha_k$.

Therefore, we set

$$A_i = \sum_{j=i}^{k} c_{ij} \alpha_j, \tag{7}$$

which yields, together with (6),

$$\sum_{j=i}^{k} c_{ij}\alpha_j - \sum_{m=i}^{k} \beta_{im} \left(\sum_{j=m}^{k} c_{mj} \alpha_j \right) - \alpha_i = 0.$$

Thus, for $0 \leq i \leq j \leq k$, we have

$$c_{jj} = -\frac{1}{2}$$

$$c_{ij} = -\frac{1}{2} \left(\sum_{m=i+1}^{j} \beta_{im} c_{mj} \right) \text{ for } i < j.$$

Therefore the particular solution $S_n^{(p)}$ of (4), we obtain

$$\begin{aligned} S_n^{(p)} &= \sum_{i=0}^{k} \left(\sum_{j=i}^{k} c_{ij} \alpha_j \right) n^i \\ &= \sum_{j=0}^{k} \left(\sum_{i=0}^{j} c_{ij} n^i \right) \alpha_j. \end{aligned}$$

Finally, the recurrence relation (4) has the solution

$$\begin{aligned} S_n &= S_n^{(h)} + S_n^{(p)} \\ &= c_1 2^n + c_2(-1)^n + \sum_{j=0}^{k}\left(\sum_{i=0}^{j} c_{ij} n^i\right)\alpha_j. \end{aligned}$$

The initial conditions: $S_0 = 1; S_1 = 1$, give

$$c_1 = \frac{2}{3} - \frac{1}{3}\left(2\Lambda_k - \Psi_k\right)$$

$$c_2 = \frac{1}{3} - \frac{1}{3}\left(\Lambda_k + \Psi_k\right),$$

where

$$\Lambda_k \quad = \textstyle\sum_{j=0}^{k} c_{0j}\alpha_j$$

and

$$\Psi_k = \begin{cases} 0, & \text{if } k = 0; \\ -\sum_{j=1}^{k}\left(\sum_{i=1}^{j} c_{ij}\right)\alpha_j, & \text{if } k > 0. \end{cases}$$

Since $B_n = \dfrac{2^{n+1} + (-1)^n}{3}$, S_n can be written as

$$S_n = (1 - \Lambda_k)B_n + \Psi_k B_{n-1} + \sum_{j=0}^{k} p_j(n)\alpha_j,$$

where

$$p_j(n) = \sum_{i=0}^{j} c_{ij} n^i.$$

The proof of the Theorem is now complete.

§3. A generalization of the Fibonacci and Beta-nacci sequences

In this section, we consider a generalization of the Fibonacci and Beta-nacci sequences. First, we define the Beta - Fibonacci sequence as follows:

Definition. Let r be a non-negative integer such that $r \geq 0$. Define the Beta-Fibonacci sequence $\{T_n\}_{n=0}^{\infty}$ as shown:

$$T_n = T_{n-1} + 2^r T_{n-2} \qquad \text{for all } n \geq 2, \tag{8}$$

with initial conditions

$$T_0 = 1; \quad T_1 = 1.$$

When $r = 0$, then the sequence $\{T_n\}_{n=0}^{\infty}$ is reduced to the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and when $r = 1$, the sequence $\{T_n\}_{n=0}^{\infty}$ is reduced to the Beta-nacci sequence $\{B_n\}_{n=0}^{\infty}$.

The general term of the Beta-Fibonacci sequence is

$$T_n = \frac{1}{\sqrt{1+2^{r+2}}}\left(\phi_1^{n+1} - \phi_2^{n+1}\right),$$

where

$$\phi_1 = \frac{1}{2}(1+\sqrt{1+2^{r+2}}) \quad \text{and} \quad \phi_2 = \frac{1}{2}(1-\sqrt{1+2^{r+2}})$$

Now we define a generalization of the Beta-Fibonacci sequence, we call generalized BF-nacci sequence.

Definition. Let r be a non-negative integer such that $r \geq 0$. For any non-negative integer k and any real numbers $\alpha_0, \alpha_1, \cdots, \alpha_k$, a generalized BF-nacci sequence $\{R_n\}_{n=0}^{\infty}$ is defined recurrently by

$$R_n = R_{n-1} + 2^r R_{n-2} + \sum_{j=0}^{k} \alpha_j n^j, \qquad \text{for all } n \geq 2, \tag{9}$$

with initial conditions

$$R_0 = 1; \quad R_1 = 1.$$

Note that if we take $r = 0$ in the definition, the sequence $\{R_n\}_{n=0}^{\infty}$ is reduced to the generalization of the Fibonacci sequence $\{G_n\}_{n=0}^{\infty}$ as shown in [1]. Also when $r = 1$, the sequence $\{R_n\}_{n=0}^{\infty}$ is reduced to the generalization of the Beta - nacci sequence $\{S_n\}_{n=0}^{\infty}$.

Theorem 2. The solution of (9) can be express as

$$R_n = (1-\Lambda_k)R_n + \Psi_k R_{n-1} + \sum_{j=0}^{k} p_j(n)\alpha_j, \tag{10}$$

where

(i) Λ_k is a linear combination of $\alpha_0, \alpha_1, \cdots, \alpha_k$;

(ii) Ψ_k is a linear combination of $\alpha_1, \cdots, \alpha_k$;

(iii) for each j ($0 \leq j \leq k$), $p_j(n)$ is a polynomail of degree j.

Proof of Theorem 2. As usual the solution $R_n^{(h)}$ of the homogeneous equation corresponding to (9) is

$$R_n^{(h)} = c_1\phi_1^n + c_2\phi_2^n,$$

where

$$\phi_1 = \frac{1}{2}(1+\sqrt{1+2^{r+2}}) \quad \text{and} \quad \phi_2 = \frac{1}{2}(1-\sqrt{1+2^{r+2}}).$$

Next, we find the particular solution of (9) . We set $R_n^{(p)} = \sum_{i=0}^{k} A_i n^i$, which yields

$$\sum_{i=0}^{k} A_i n^i = \sum_{i=0}^{k} A_i (n-1)^i + 2^r \sum_{i=0}^{k} A_i (n-2)^i + \sum_{i=0}^{k} \alpha_i n^i.$$

Thus, for each $i = 0, 1, \cdots, k$, we have

$$A_i - \sum_{m=i}^{k} \beta_{im} A_m - \alpha_i = 0 \tag{11}$$

with, for $m \geq i$,

$$\beta_{im} = \binom{m}{i} (-1)^{m-i} (1 + 2^{m-i+r}).$$

From the recurrence relation (11), we can successively determine $A_k, A_{k-1}, \cdots, A_0$: the coefficient A_i is a linear combination of $\alpha_i, \alpha_{i+1}, \cdots, \alpha_k$.

Therefore, we set

$$A_i = \sum_{j=i}^{k} c_{ij} \alpha_j, \tag{12}$$

which yields, together with (11),

$$\sum_{j=i}^{k} c_{ij} \alpha_j - \sum_{m=i}^{k} \beta_{im} \left(\sum_{j=m}^{k} c_{mj} \alpha_j \right) - \alpha_i = 0.$$

Thus, for $0 \leq i \leq j \leq k$, we have

$$c_{jj} = -\frac{1}{2^r}$$

$$c_{ij} = -\frac{1}{2^r} \left(\sum_{m=i+1}^{j} \beta_{im} c_{mj} \right) \qquad \text{for } i < j.$$

Hence, for the particular solution $R_n^{(p)}$ of (9), we obtain

$$\begin{aligned} R_n^{(p)} &= \sum_{i=0}^{k} \left(\sum_{j=i}^{k} c_{ij} \alpha_j \right) n^i \\ &= \sum_{j=0}^{k} \left(\sum_{i=0}^{j} c_{ij} n^i \right) \alpha_j. \end{aligned}$$

Finally, the recurrence relation (9) has the solution

$$\begin{aligned} R_n &= R_n^{(h)} + R_n^{(p)} \\ &= c_1\phi_1^n + c_2\phi_2^n + \sum_{j=0}^{k}\Big(\sum_{i=0}^{j} c_{ij}n^i\Big)\alpha_j. \end{aligned}$$

The initial conditions: $R_0 = 1;\quad R_1 = 1$, give

$$c_1 = \frac{1}{\sqrt{1+2^{r+2}}}\Big((1-R_0^{(p)})\phi_1 + R_0^{(p)} - R_1^{(p)}\Big)$$

$$c_2 = -\frac{1}{\sqrt{1+2^{r+2}}}\Big((1-R_0^{(p)})\phi_2 + R_0^{(p)} - R_1^{(p)}\Big)$$

Since $T_n = \frac{1}{\sqrt{1+2^{r+2}}}\Big(\phi_1^{n+1} - \phi_2^{n+1}\Big)$, R_n can be written as

$$R_n = (1-R_0^{(p)})T_n + (R_0^{(p)} - R_1^{(p)})T_{n-1} + \sum_{j=0}^{k} p_j(n)\alpha_j,$$

where

$$p_j(n) = \sum_{i=0}^{j} c_{ij}n^i.$$

Hence the proof is complete.

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The product of divisors minimum and maximum functions

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Abstract Let $T(n)$ denote the product of divisors of the positive integer n. We introduce and study some basic properties involving two functions, which are the minimum, resp. the maximum of certain integers connected with the divisors of T(n).

Keywords Arithmetic functions, product of divisors of an integer.

1. Let $T(n) = \prod_{i|n} i$ denote the product of all divisors of n. The product-of-divisors minimum, resp. maximum functions will be defined by

$$\mathcal{T}(n) = \min\{k \geq 1 : \ n|T(k)\} \tag{1}$$

and

$$\mathcal{T}_*(n) = \max\{k \geq 1 : \ T(k)|n\}. \tag{2}$$

There are particular cases of the functions F_f^A, G_g^A defined by

$$F_f^A(n) = \min\{k \in A : \ n|f(k)\}, \tag{3}$$

and its "dual"

$$G_g^A(n) = \max\{k \in A : \ g(k)|n\}, \tag{4}$$

where $A \subset \mathbb{N}^*$ is a given set, and $f, g : \mathbb{N}^* \to \mathbb{N}$ are given functions, introduced in [8] and [9]. For $A = \mathbb{N}^*$, $f(k) = g(k) = k!$ one obtains the Smarandache function $S(n)$, and its dual $S_*(n)$, given by

$$S(n) = \min\{k \geq 1 : \ n|k!\} \tag{5}$$

and

$$S_*(n) = \max\{k \geq 1 : \ k!|n\}. \tag{6}$$

The function $S_*(n)$ has been studied in [8], [9], [4], [1], [3]. For $A = \mathbb{N}^*$, $f(k) = g(k) = \varphi(k)$, one obtains the Euler minimum, resp. maximum functions

$$E(n) = \min\{k \geq 1 : \ n|\varphi(k)\} \tag{7}$$

studied in [6], [8], [13], resp., its dual

$$E_*(n) = \max\{k \geq 1 : \ \varphi(k)|n\}, \tag{8}$$

studied in [13].

For $A = \mathbb{N}^*$, $f(k) = g(k) = S(k)$ one has the Smarandache minimum and maximum functions

$$S_{min}(n) = \min\{k \geq 1: \ n|S(k)\}, \tag{9}$$

$$S_{max}(n) = \max\{k \geq 1: \ S(k)|n\}, \tag{10}$$

introduced, and studied in [15]. The divisor minimum function

$$D(n) = \min\{k \geq 1: \ n|d(k)\} \tag{11}$$

(where $d(k)$ is the number of divisors of k) appears in [14], while the sum-of-divisors minimum and maximum functions

$$\Sigma(n) = \min\{k \geq 1: \ n|\sigma(k)\} \tag{12}$$

$$\Sigma_*(n) = \max\{k \geq 1: \ \sigma(k)|n\} \tag{13}$$

have been recently studied in [16].

For functions $Q(n), Q_1(n)$ obtained from (3) for $f(k) = k!$ and $A =$ set of perfect squares, resp. $A =$ set of squarefree numbers, see [10].

2. The aim of this note is to study some properties of the functions $\mathcal{T}(n)$ and $\mathcal{T}_*(n)$ given by (1) and (2). We note that properties of $T(n)$ in connection with "multiplicatively perfect numbers" have been introduced in [11]. For other asymptotic properties of $T(n)$, see [7]. For divisibility properties of $T(\sigma(n))$ with $T(n)$, see [5]. For asymptotic results of sums of type $\sum\limits_{n\leq x} \frac{1}{T(n)}$, see [17].

A divisor i of n is called "unitary" if $\left(i, \frac{n}{i}\right) = 1$. Let $T^*(n)$ be the product of unitary divisors of n. For similar results to [11] for $T^*(n)$, or $T^{**}(n)$ (i.e. the product of "bi-unitary" divisors of n), see [2]. The product of "exponential" divisors $T_e(n)$ is introduced in paper [12]. Clearly, one can introduce functions of type (1) and (2) for $T(n)$ replaced with one of the above functions $T^*(n), T^{**}, T_e(n)$, but these functions will be studied in another paper.

3. The following auxiliary result will be important in what follows.

Lemma 1.

$$T(n) = n^{d(n)/2}, \tag{14}$$

where $d(n)$ is the number of divisors of n.

Proof. This is well-known, see e.g. [11].

Lemma 2.

$$T(a)|T(b), \quad \text{if } a|b. \tag{15}$$

Proof. If $a|b$, then for any $d|a$ one has $d|b$, so $T(a)|T(b)$. Reciprocally, if $T(a)|T(b)$, let $\gamma_p(a)$ be the exponent of the prime in a. Clearly, if $p|a$, then $p|b$, otherwise $T(a)|T(b)$ is impossible. If $p^{\gamma_p(b)}\|b$, then we must have $\gamma_p(a) \leq \gamma_p(b)$. Writing this fact for all prime divisors of a, we get $a|b$.

Theorem 1. If n is squarefree, then

$$\mathcal{T}(n) = n. \tag{16}$$

Proof. Let $n = p_1p_2 \dots p_r$, where p_i $(i = \overline{1,r})$ are distinct primes. The relation $p_1p_2 \dots p_r | T(k)$ gives $p_i | T(k)$, so there is a $d|k$, so that $p_i|d$. But then $p_i|k$ for all $i = \overline{1,r}$, thus $p_1p_2 \dots p_r = n|k$. Since $p_1p_2 \dots p_k | T(p_1p_2 \dots p_k)$, the least k is exactly $p_1p_2 \dots p_r$, proving (16).

Remark. Thus, if p is a prime, $\mathcal{T}(p) = p$; if $p < q$ are primes, then $\mathcal{T}(pq) = pq$, etc.

Theorem 2. If $a|b$, $a \neq b$ and b is squarefree, then

$$\mathcal{T}(ab) = b. \tag{17}$$

Proof. If $a|b$, $a \neq b$, then clearly $T(b) = \prod_{d|b} d$ is divisible by ab, so $\mathcal{T}(ab) \leq b$. Reciprocally, if $ab|T(k)$, let $p|b$ a prime divisor of b. Then $p|T(k)$, so (see the proof of Theorem 1) $p|k$. But b being squarefree (i.e. a product of distinct primes), this implies $b|k$. The least such k is clearly $k = b$.

For example, $\mathcal{T}(12) = \mathcal{T}(2 \cdot 6) = 6$, $\mathcal{T}(18) = \mathcal{T}(3 \cdot 6) = 6$, $\mathcal{T}(20) = \mathcal{T}(2 \cdot 10) = 10$.

Theorem 3. $\mathcal{T}(T(n)) = n$ for all $n \geq 1$. (18)

Proof. Let $T(n)|T(k)$. Then by (15) one can write $n|k$. The least k with this property is $k = n$, proving relation (18).

Theorem 4. Let p_i $(i = \overline{1,r})$ be distinct primes, and $\alpha_i \geq 1$ positive integers. Then

$$\max\left\{\mathcal{T}\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) : \ i = \overline{1,r}\right\} \leq \mathcal{T}\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq$$

$$\leq l.c.m.[\mathcal{T}(p_1^{\alpha_1}), \dots, \mathcal{T}(p_r^{\alpha_r})]. \tag{19}$$

Proof. In [13] it is proved that for $A = \mathbb{N}^*$, and any function f such that $F_f^{\mathbb{N}^*}(n) = F_f(n)$ is well defined, one has

$$\max\{F_f(p_i^{\alpha_i}) : \ i = \overline{1,r}\} \leq F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right). \tag{20}$$

On the other hand, if f satisfies the property

$$a|b \Longrightarrow f(a)|f(b)(a, b \geq 1), \tag{21}$$

then

$$F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq l.c.m.[F_f(p_1^{\alpha_1}), \dots, F_f(p_r^{\alpha_r})]. \tag{22}$$

By Lemma 2, (21) is true for $f(a) = T(a)$, and by using (20), (22), relation (19) follows.

Theorem 5.

$$\mathcal{T}(2^n) = 2^\alpha, \tag{23}$$

where α is the least positive integer such that

$$\frac{\alpha(\alpha+1)}{2} \geq n. \tag{24}$$

Proof. By (14), $2^n|T(k)$ iff $2^n|k^{d(k)/2}$. Let $k = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, when $d(k) = (\alpha_1+1)\dots(\alpha_r+1)$. Since $2^{2n}|k^{d(k)} = p_1^{\alpha_1(\alpha_1+1)\dots(\alpha_r+1)} \dots p_r^{\alpha_r(\alpha_1+1)\dots(\alpha_r+1)}$ (let $p_1 < p_2 < \dots < p_r$), clearly $p_1 = 2$

and the least k is when $\alpha_2 = \cdots = \alpha_r = 0$ and α_1 is the least positive integer with $2n \leq \alpha_1(\alpha_1 + 1)$. This proves (23), with (24).

For example, $\mathcal{T}(2^2) = 4$, since $\alpha = 2$, $\mathcal{T}(2^3) = 4$ again, $\mathcal{T}(2^4) = 8$ since $\alpha = 3$, etc.

For odd prime powers, the things are more complicated. For example, for 3^n one has:

Theorem 6.

$$\mathcal{T}(3^n) = \min\{3^{\alpha_1}, 2 \cdot 3^{\alpha_2}\}, \tag{25}$$

where α_1 is the least positive integer such that $\frac{\alpha_1(\alpha_1+1)}{2} \geq n$, and α_2 is the least positive integer such that $\alpha_2(\alpha_2 + 1) \geq n$.

Proof. As in the proof of Theorem 5,

$$3^{2n} | p_1^{\alpha_1(\alpha_1+1)\dots(\alpha_r+1)} \cdot p_2^{\alpha_2(\alpha_1+1)\dots(\alpha_1+1)} \dots p_r^{\alpha_r(\alpha_1+1)\dots(\alpha_r+1)},$$

where $p_1 < p_2 < \cdots < p_r$, so we can distinguish two cases:

a) $p_1 = 2,\ p_2 = 3,\ p_3 \geq 5$;

b) $p_1 = 3,\ p_2 \geq 5$.

Then $k = 2^{\alpha_1} \cdot 3^{\alpha_2} \dots p_r^{\alpha_r} \geq 2^{\alpha_1} \cdot 3^{\alpha_2}$ in case a), and $k \geq 3^{\alpha_1}$ in case b). So for the least k we must have $\alpha_2(\alpha_1 + 1)(\alpha_2 + 1) \geq 2n$ with $\alpha_1 = 1$ in case a), and $\alpha_1(\alpha_1 + 1) \geq 2n$ in case b). Therefore $\frac{\alpha_1(\alpha_1+1)}{2} \geq n$ and $\alpha_2(\alpha_2 + 1) \geq n$, and we must select k with the least of 3^{α_1} and $2^1 \cdot 3^{\alpha_2}$, so Theorem 6 follows.

For example, $\mathcal{T}(3^2) = 6$ since for $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, and $\min\{2 \cdot 3^1, 3^2\} = 6$; $\mathcal{T}(3^3) = 9$ since for $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 2$ and $\min\{2 \cdot 3^2, 3^2\} = 9$.

Theorem 7. Let $f : [1, \infty) \to [0, \infty)$ be given by $f(x) = \sqrt{x} \log x$. Then

$$f^{-1}(\log n) < \mathcal{T}(n) \leq n, \tag{26}$$

for all $n \geq 1$, where f^{-1} denotes the inverse function of f.

Proof. Since $n|T(n)$, the right side of (26) follows by definition (1) of $\mathcal{T}(n)$. On the other hand, by the known inequality $d(k) < 2\sqrt{k}$, and Lemma 1 (see (14)) we get $T(k) < k^{\sqrt{k}}$, so $\log T(k) < \sqrt{k} \log k = f(k)$. Since $n|T(k)$ implies $n \leq T(k)$, so $\log n \leq \log T(k) < f(k)$, and the function f being strictly increasing and continuous, by the bijectivity of f, the left side of (26) follows.

4. The function $\mathcal{T}_*(n)$ given by (2) differs in many aspects from $\mathcal{T}(n)$. The first such property is:

Theorem 8. $\mathcal{T}_*(n) \leq n$ for all n, with equality only if $n = 1$ or $n =$ prime.

Proof. If $T(k)|n$, then $T(k) \leq n$. But $T(k) \geq k$, so $k \leq n$, and the inequality follows.

Let us now suppose that for $n > 1$, $\mathcal{T}_*(n) = n$. Then $T(n)|n$, by definition 2. On the other hand, clearly $n|T(n)$, so $T(n) = n$. This is possible only when $n =$ prime.

Remark. Therefore the equality

$$\mathcal{T}_*(n) = n (n > 1)$$

is a characterization of the prime numbers.

Lemma 3. Let $p_1, \dots, p_r$ be given distinct primes ($r \geq 1$). Then the equation

$$T(k) = p_1 p_2 \dots p_r$$

is solvable if $r = 1$.

Proof. Since $p_i | T(k)$, we get $p_i | k$ for all $i = \overline{1, r}$. Thus $p_1 \dots p_r | k$, and Lemma 2 implies $T(p_1 \dots p_r) | T(k) = p_1 \dots p_r$. Since $p_1 \dots p_r | T(p_1 \dots p_r)$, we have $T(p_1 \dots p_r) = p_1 \dots p_r$, which by Theorem 8 is possible only if $r = 1$.

Theorem 9. Let $P(n)$ denote the greatest prime factor of $n > 1$. If n is squarefree, then

$$\mathcal{T}_*(n) = P(n). \tag{27}$$

Proof. Let $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots < p_r$. If $T(k) | (p_1 \dots p_r)$, then clearly $T(k) \in \{1, p_1, \dots, p_r, p_1 p_2, \dots, p_1 p_2 \dots p_r\}$. By Lemma 3 we cannot have

$$T(k) \in \{p_1 p_2, \dots, p_1 p_2 \dots p_r\},$$

so $T(k) \in \{1, p_1, \dots, p_r\}$, when $k \in \{1, p_1, \dots, p_r\}$. The greatest k is $p_r = P(n)$.

Remark. Therefore $\mathcal{T}_*(pq) = q$ for $p < q$. For example, $\mathcal{T}_*(2 \cdot 7) = 7$, $\mathcal{T}_*(3 \cdot 5) = 5$, $\mathcal{T}_*(3 \cdot 7) = 7$, $\mathcal{T}_*(2 \cdot 11) = 11$, etc.

Theorem 10.

$$\mathcal{T}_*(p^n) = p^\alpha (p = \text{prime}), \tag{28}$$

where α is the greatest integer with the property

$$\frac{\alpha(\alpha+1)}{2} \leq n. \tag{29}$$

Proof. If $T(k) | p^n$, then $T(k) = p^m$ for $m \leq n$. Let q be a prime divisor of k. Then $q = T(q) | T(k) = 2^m$ implies $q = p$, so $k = p^\alpha$. But then $T(k) = p^{\alpha(\alpha+1)/2}$ with α the greatest number such that $\alpha(\alpha+1)/2 \leq n$, which finishes the proof of (28).

For example, $\mathcal{T}_*(4) = 2$, since $\frac{\alpha(\alpha+1)}{2} \leq 2$ gives $\alpha_{max} = 1$.

$\mathcal{T}_*(16) = 4$, since $\frac{\alpha(\alpha+1)}{2} \leq 4$ is satisfied with $\alpha_{max} = 2$.

$\mathcal{T}_*(9) = 3$, and $\mathcal{T}_*(27) = 9$ since $\frac{\alpha(\alpha+1)}{2} \leq 3$ with $\alpha_{max} = 2$.

Theorem 11. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^2 q) = \max\{p, q\}. \tag{30}$$

Proof. If $T(k) | p^2 q$, then $T(k) \in \{1, p, q, p^2, pq, p^2 q\}$. The equations $T(k) = p^2$, $T(k) = pq$, $T(k) = p^2 q$ are impossible. For example, for the first equation, this can be proved as follows. By $p | T(k)$ one has $p | k$, so $k = pm$. Then $p(pm)$ are in $T(k)$, so $m = 1$. But then $T(k) = p \neq p^2$. For the last equation, $k = (pq)m$ and $pqm(pm)(qm)(pqm)$ are in $T(k)$, which is impossible.

Theorem 12. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^3 q) = \max\{p^2, q\}. \tag{31}$$

Proof. As above, $T(k) \in \{1, p, q, pq, p^2 q, p^3 q, p^2, p^3\}$ and $T(k) \in \{pq, p^2 q, p^3 q, p^2\}$ are impossible. But $T(k) = p^3$ by Lemma 1 gives $k^{d(k)} = p^6$, so $k = p^m$, when $d(k) = m + 1$. This gives $m(m+1) = 6$, so $m = 2$. Thus $k = p^2$. Since $p < p^2$ the result follows.

Remark. The equation

$$T(k) = p^s \tag{32}$$

can be solved only if $k^{d(k)} = p^{2s}$, so $k = p^m$ and we get $m(m+1) = 2s$. Therefore $k = p^m$, with $m(m+1) = 2s$, if this is solvable. If s is not a **triangular number**, this is impossible.

Theorem 13. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^s q) = \begin{cases} \max\{p, q\}, & \text{if } s \text{ is not a triangular number}, \\ \max\{p^n, q\}, & \text{if } s = \frac{m(m+1)}{2}. \end{cases}$$

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Generalized random stability of Jensen type mapping

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Abstract In this paper, we consider Jensen type mapping in the setting of generalized random normed spaces. We generalize a Hyers-Ulam stability result in the framework of classical normed spaces.

Keywords Jensen type mapping, generalized random normed space.

§1. Introduction

In 1941 D.H. Hyers [5] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 1951 D.G. Bourgin [2] was the second author to treat the Ulam stability problem for additive mappings. In 1978 P.M. Gruber [4] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types.

We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [10] used a stability property of the functional equation

$$f(x-y)+f(x+y)=2f(x) \tag{0.1}$$

to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials. In 2003–2006 J.M. Rassias and M.J. Rassias [6] and J.M. Rassias [7] solved the above Ulam problem for Jensen and Jensen type mappings. In this paper we consider the stability of Jensen type mapping in the setting of intuitionistic fuzzy normed spaces.

§2. Preliminaries

In the sequel, we shall adopt the usual terminology, notations and conventions of the theory of intuitionistic random normed spaces as in [8].

Definition 1. A measure distribution function is a function $\mu : \mathbb{R} \to [0,1]$ which is left continuous on $\mathbb{R}$, non-decreasing and $\inf_{t\in\mathbb{R}} \mu(t) = 0$, $\sup_{t\in\mathbb{R}} \mu(t) = 1$.

We will denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\mu : X \longrightarrow D$ is called a *probabilistic measure* on X and $\mu(x)$ is denoted by μ_x.

Definition 2. A non-measure distribution function is a function $\nu : \mathbb{R} \to [0,1]$ which is right continuous on $\mathbb{R}$, non-increasing and $\inf_{t\in\mathbb{R}} \nu(t) = 1$, $\sup_{t\in\mathbb{R}} \nu(t) = 0$.

We will denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ 0 & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\nu : X \longrightarrow B$ is called a probabilistic non-measure on X and $\nu(x)$ is denoted by ν_x.

lemma 1. [1,3] Consider the set L^* and operation $\leq_{L^*}$ defined by:

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1,\ x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying $T(1,x) = 1 * x = x$ for all $x \in [0,1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \longrightarrow [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 3. [3] A triangular norm (t–norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

(a) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition);

(b) $(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$ (commutativity);

(c) $(\forall (x,y,z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z))$ (associativity);

(d) $(\forall (x,x',y,y') \in (L^*)^4)(x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x',y'))$ (monotonicity).

If $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}, then $\mathcal{T}$ is said to be a continuous t–norm.

Definition 4. [3] A continuous t–norm $\mathcal{T}$ on L^* is said to be continuous t–representable if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a,b) = (a_1b_1, \min\{a_2+b_2, 1\})$$

and

$$\mathbf{M}(a,b) = (\min\{a_1,b_1\}, \max\{a_2,b_2\})$$

for all $a=(a_1,a_2)$, $b=(b_1,b_2)\in L^*$ are continuous t–representable.

Now, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1=\mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)},\cdots,x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)},\cdots,x^{(n)}),x^{(n+1)}), \quad \forall n\geq 2,\ x^{(i)}\in L^*.$$

Definition 5. A negator on L^* is any decreasing mapping $\mathcal{N}: L^*\longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*})=1_{L^*}$ and $\mathcal{N}(1_{L^*})=0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x))=x$ for all $x\in L^*$, then $\mathcal{N}$ is called an involutive negator. A *negator* on $[0,1]$ is a decreasing mapping $N:[0,1]\longrightarrow[0,1]$ satisfying $N(0)=1$ and $N(1)=0$. N_s denotes the standard negator on $[0,1]$ defined by

$$N_s(x)=1-x, \quad \forall x\in[0,1].$$

Definition 6. Let μ and ν be measure and non-measure distribution functions from $X\times(0,+\infty)$ to $[0,1]$ such that $\mu_x(t)+\nu_x(t)\leq 1$ for all $x\in X$ and $t>0$. The triple $(X,\mathcal{P}_{\mu,\nu},\mathcal{T})$ is said to be an intuitionistic random normed space (briefly IRN-space) if X is a vector space, $\mathcal{T}$ is a continuous t–representable and $\mathcal{P}_{\mu,\nu}$ is a mapping $X\times(0,+\infty)\to L^*$ satisfying the following conditions: for all $x,y\in X$ and $t,s>0$,

(a) $\mathcal{P}_{\mu,\nu}(x,0)=0_{L^*}$;

(b) $\mathcal{P}_{\mu,\nu}(x,t)=1_{L^*}$ if and only if $x=0$;

(c) $\mathcal{P}_{\mu,\nu}(\alpha x,t)=\mathcal{P}_{\mu,\nu}(x,\frac{t}{|\alpha|})$ for all $\alpha\neq 0$;

(d) $\mathcal{P}_{\mu,\nu}(x+y,t+s)\geq_{L^*}\mathcal{T}(\mathcal{P}_{\mu,\nu}(x,t),\mathcal{P}_{\mu,\nu}(y,s))$.

In this case, $\mathcal{P}_{\mu,\nu}$ is called an intuitionistic random norm. Here,

$$\mathcal{P}_{\mu,\nu}(x,t)=(\mu_x(t),\nu_x(t)).$$

Example 1. Let $(X,\|\cdot\|)$ be a normed space. Let $\mathcal{T}(a,b)=(a_1b_1,\min(a_2+b_2,1))$ for all $a=(a_1,a_2)$, $b=(b_1,b_2)\in L^*$ and μ,ν be measure and non-measure distribution functions defined by

$$\mathcal{P}_{\mu,\nu}(x,t)=(\mu_x(t),\nu_x(t))=\Big(\frac{t}{t+\|x\|},\frac{\|x\|}{t+\|x\|}\Big), \quad \forall t\in\mathbb{R}^+.$$

Then $(X,\mathcal{P}_{\mu,\nu},\mathcal{T})$ is an IRN-space.

Definition 7. (1) A sequence $\{x_n\}$ in an IRN-space $(X,\mathcal{P}_{\mu,\nu},\mathcal{T})$ is called a Cauchy sequence if, for any $\varepsilon>0$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that

$$\mathcal{P}_{\mu,\nu}(x_n-x_m,t)>_{L^*}(N_s(\varepsilon),\varepsilon), \quad \forall n,m\geq n_0,$$

where N_s is the standard negator.

(2) The sequence $\{x_n\}$ is said to be convergent to a point $x\in X$ (denoted by $x_n\xrightarrow{\mathcal{P}_{\mu,\nu}}x$) if $\mathcal{P}_{\mu,\nu}(x_n-x,t)\longrightarrow 1_{L^*}$ as $n\longrightarrow\infty$ for every $t>0$.

(3) An IRN-space $(X,\mathcal{P}_{\mu,\nu},\mathcal{T})$ is said to be complete if every Cauchy sequence in X is convergent to a point $x\in X$.

§3. Stability results

Theorem 1. Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IRN-space, $\varphi : X \times X \longrightarrow Z$ be a function such that for some $0 < \alpha < 2$,

$$\mathcal{P}'_{\mu,\nu}(\varphi(2x, 2x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\alpha\varphi(x, x), t) \quad (x, \in X, t > 0) \tag{0.2}$$

and $\lim_{n\to\infty} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), 2^n t) = 1_{L^*}$ for all $x, y \in X$ and $t > 0$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IRN-space. If $f : X \to Y$ is a mapping such that

$$\begin{aligned} &\mathcal{P}_{\mu,\nu}(f(x+y) - f(x-y) - 2f(y), t) \\ \geq_{L^*} \quad &\mathcal{P}'_{\mu,\nu}(\varphi(x, y), t) \quad (x, y \in X, t > 0) \end{aligned} \tag{0.3}$$

and $f(0) = 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, y), (2 - \alpha)t)). \tag{0.4}$$

Proof. Putting $y = x$ in (0.3) we get

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2x)}{2} - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, x), 2t) \quad (x \in X, t > 0). \tag{0.5}$$

Replacing x by $2^n x$ in (0.5), and using (0.2) we obtain

$$\begin{aligned} \mathcal{P}_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, t\right) \quad &\geq_{L^*} \quad \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n x), 2 \times 2^n t) \\ &\geq_{L^*} \quad \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{2 \times 2^n}{\alpha^n}\right). \end{aligned} \tag{0.6}$$

Since $\frac{f(2^n x)}{2^n} - f(x) = \sum_{k=0}^{n-1}(\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k})$, by (0.6) we have

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), t\sum_{k=0}^{n-1}\frac{\alpha^k}{2 \times 2^k}\right) \geq_{L^*} \mathbf{M}_{k=0}^{n-1}\left(\mathcal{P}'_{\mu,\nu}(\varphi(x, x), t)\right) = \mathcal{P}'_{\mu,\nu}(\varphi(x, x), t),$$

that is

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 2^k}}\right). \tag{0.7}$$

By replacing x with $2^m x$ in (0.7) we observe that:

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, t\right) \geq \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{2\times 2^k}}\right). \tag{0.8}$$

Then $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$. Since $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ is a complete IRN-space this sequence convergent to some point $A(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (0.8) to obtain

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 2^k}}\right), \tag{0.9}$$

and so for every $\delta > 0$ we have that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(A(x)-f(x),t+\delta) &\geq_{L^*} \mathbf{M}\left(\mathcal{P}_{\mu,\nu}\left(A(x)-\frac{f(2^n x)}{2^n},\delta\right),\mathcal{P}_{\mu,\nu}\left(f(x)-\frac{f(2^n x)}{2^n},t\right)\right) \quad (0.10)\\ &\geq_{L^*} \mathbf{M}\left(\mathcal{P}_{\mu,\nu}\left(A(x)-\frac{f(2^n x)}{2^n},\delta\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{2\times 2^k}}\right)\right).\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ and using (0.10) we get

$$\mathcal{P}_{\mu,\nu}(A(x)-f(x),t+\delta) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x),t(2-\alpha)). \tag{0.11}$$

Since δ was arbitrary, by taking $\delta \to 0$ in (0.11) we get

$$\mathcal{P}_{\mu,\nu}(A(x)-f(x),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x,x),t(2-\alpha)).$$

Replacing x, y by $2^n x, 2^n y$ in (0.3) to get

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(\frac{f(2^n(x+y))}{2^n}+\frac{f(2^n(x-y))}{2^n}-\frac{2f(2^n y)}{2^n},t)\\ \geq_{L^*}\ &\mathcal{P}'_{\mu,\nu}(\varphi(2^n x,2^n y),2^n t),\end{aligned} \tag{0.12}$$

for all $x, y \in X$ and for all $t > 0$. Since $\lim_{n\longrightarrow\infty} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), 2^n t) = 1_{L^*}$ we conclude that A fulfills (0.1). To Prove the uniqueness of the additive function A, assume that there exists an additive function $A' : X \longrightarrow Y$ which satisfies (0.4). Fix $x \in X$. Clearly $A(2^n x) = 2^n A(x)$ and $A'(2^n x) = 2^n A(x)$ for all $n \in \mathbb{N}$. It follows from (0.4) that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(A(x)-A'(x),t) &= \mathcal{P}_{\mu,\nu}\left(\frac{A(2^n x)}{2^n}-\frac{A'(2^n x)}{2^n},t\right)\\ &\geq_{L^*} \mathbf{M}\left\{\mathcal{P}_{\mu,\nu}\left(\frac{A(2^n x)}{2^n}-\frac{f(2^n x)}{2^n},\frac{t}{2}\right),\mathcal{P}_{\mu,\nu}\left(\frac{A'(2^n x)}{2^n}-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)\right\}\\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(2^n x,2^n x),2^n(2-\alpha)\frac{t}{2}\right)\\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x,x),\frac{2^n(2-\alpha)\frac{t}{2}}{\alpha^n}\right).\end{aligned}$$

Since $\lim_{n\to\infty} \frac{27^n(27-\alpha)t}{2\alpha^n} = \infty$, we get $\lim_{n\to\infty} \mathcal{P}'_{\mu,\nu}(\varphi(x,0), \frac{27^n(27-\alpha)t}{2\alpha^n}) = 1_{L^*}$. Therefore $\mathcal{P}_{\mu,\nu}(A(x)-A'(x),t) = 1$ for all $t > 0$, whence $A(x) = A'(x)$.

Corollary 1. Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IRN-space, $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IRN-space, p, q be nonnegative real numbers and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that

$$\mathcal{P}_{\mu,\nu}(f(x+y)+f(x-y)-2f(y),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p+\|y\|^q)z_0,t), \tag{0.13}$$

$x, y \in X$, $t > 0$, $f(0) = 0$ and $p, q < 1$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x)-A(x),t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^p z_0,(2-2^p)t)). \tag{0.14}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 2^p$.

Corollary 2. Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IRN-space, $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IRN-space and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that

$$\mathcal{P}_{\mu,\nu}(f(x+y) + f(x-y) - 2f(y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varepsilon z_0, t) \tag{0.15}$$

$x, y \in X$, $t > 0$, $f(0) = 0$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varepsilon z_0, t). \tag{0.16}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 1$.

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On Isomorphisms of KU-algebras

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Abstract In this paper, we study homomorphisms of KU-algebras and investigate its properties. Moreover, some consequences of the relations between quotient KU-algebras and isomorphisms are shown.

Keywords Homomorphism, isomorphism, ideal, congruence, KU-algebras.

§1. Introduction

In [3], C. Prabpayak and U. Leerawat studied ideals and congruences of BCC-algebras ([1],[2]) and introduced a new algebraic structure which is called KU-algebras. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. The purpose of this paper is to derive some straightforward consequences of the relations between quotient KU-algebras and isomorphisms and also investigate some of its properties.

§2. Preliminaries

A nonempty set G with a constant 0 and a binary operation denoted by juxtaposition is called a KU-algebra if for all for all $x, y, z \in G$ the following conditions hold:

(1) $(xy)((yz)(xz)) = 0$,

(2) $0x = x$,

(3) $x0 = 0$,

(4) $xy = 0 = yx$ implies $x = y$,

for all $x, y, z \in G$.

By (1), we get $(00)((0x)(0x)) = 0$. It follows that $xx = 0$ for all $x \in G$. And if we put $y = 0$ in (1), then we obtain $z(xz) = 0$ for all $x, z \in G$.

A subset S of a KU-algebra G is called subalgebra of G if $xy \in S$ whenever $x, y \in S$. A non-empty subset A of a KU-algebra G is called an ideal of G if it satisfies the following conditions:

(5) $0 \in A$,

(6) for all $x, y, z \in G$, $x(yz) \in A$ and $y \in A$ imply $xz \in A$.

Putting $x = 0$ in (6), we obtain the following: for all $x, y \in G$, $xy \in A$ and $x \in A$ imply $y \in A$.

On KU-algebra $(G, \cdot, 0)$. We define a binary relation $\leq$ on G by putting $x \leq y$ if and only if $yx = 0$. Then $(G; \leq)$ is a partially ordered set and 0 is its smallest element. Thus a KU-algebra G satisfies conditions: $(yz)(xz) \leq xy$, $0 \leq x$, $x \leq y \leq x$ implies $x = y$.

Let A be an ideal of KU-algebra G. Define the relation $\sim$ on G by $x \sim y$ if and only if $xy \in A$ and $yx \in A$. Then the relation $\sim$ is a congruence on G. And $C_0 = \{x \in G \mid x \sim 0\}$ is an ideal of G.

Let $\sim$ be a congruence relation on a KU-algebra G and let A be an ideal of G. Define A_x by $A_x = \{y \in G \mid y \sim x\} = \{y \in G \mid xy \in A, yx \in A\}$. Then the family $\{A_x : x \in G\}$ gives a partition of G which is denoted by G/A. For any $x, y \in G$, we define $A_x * A_y = A_{xy}$. Since $\sim$ has the substitution property, the operation * is well-defined. It is easily checked that $(G/A, *, A)$ is a KU-algebra.

If A is an ideal of KU-algebra G, then it is clear that $A_x = A_0 = A$ for all x in A.

Let $(G, \cdot, 0)$ and $(H, *, 0)$ be KU-algebras. A homomorphism is a map $f : G \to H$ satisfying $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in G$. An injective homomorphism is called monomorphism, a surjective homomorpism is called epimorphism and a bijective homomorphism is called isomorphism. The kernel of the homomorphism f, denoted by $\ker f$, is the set of elements of G that map to 0 in H.

§3. Results

For an ideal A of a KU-algebra G. Then the canonical mapping $f : G \to G/A$ defined by $f(x) = A_x$ is an epimorphism and $\ker f$ is an ideal of G (see in [3]).

Definition 1. Let ϕ be a mapping of a KU-algebra G into a KU-algebra H, and let $A \subseteq G$ and $B \subseteq H$. The image of A in H under ϕ is

$$\phi(A) = \{\phi(a) \mid a \in A\}$$

and the inverse image of B in G is

$$\phi^{-1}(B) = \{g \in G \mid \phi(g) \in B\}.$$

Theorem 1. Let ϕ be a homomorphism of a KU-algebras G into a KU-algebra H.

(i) If 0 is the identity in G, then $\phi(0)$ is the identity in H.

(ii) If S is a KU-subalgebra of G, then $\phi(S)$ is a KU-subalgebra of H.

(iii) If A is an ideal of G, then $\phi(A)$ is an ideal in $\phi(G)$.

(iv) If K is a KU-subalgebra of H, then $\phi^{-1}(K)$ is a KU-subalgebra of G.

(v) If B is an ideal in $\phi(G)$, then $\phi^{-1}(B)$ is an ideal in G.

(vi) If ϕ is a homomorphism from a KU-algebra G to a KU-algebra H, then ϕ is 1-1 if and only if $\ker\phi = \{0\}$.

Proof of Theorem 1.

(i) Let 0 be the identity in G and $\acute{0}$ the identity in H. Then $\phi(0)\acute{0} = \acute{0}$ and

$$\begin{aligned} \acute{0}\phi(0) &= (\phi(0)\phi(0))\phi(0) \\ &= \phi(00)\phi(0) \\ &= \phi(0)\phi(0) \\ &= \acute{0}. \end{aligned}$$

By (4), we get that $\phi(0) = \acute{0}$.

(ii) Let S be a KU-subalgebra of G. Let $x, y \in \phi(A)$. That means $x = \phi(a)$ and $y = \phi(b)$ for some a, b in G. Then $xy = \phi(a)\phi(b) = \phi(ab)$. Thus $\phi(A)$ is a KU-subalgebra of H.

(iii) Let A be an ideal of G. Clearly, $\acute{0} \in \phi(A)$. If $\phi(x)(\phi(y)\phi(z)) \in \phi(A)$ and $\phi(y) \in \phi(A)$, then $\phi(x(yz)) \in \phi(A)$ and $\phi(y) \in \phi(A)$, so $x(yz), y \in A$. Since A is an ideal, $xz \in A$. Thus $\phi(x)\phi(z) = \phi(xz) \in \phi(A)$. Hence $\phi(A)$ is an ideal of $\phi(G)$.

(iv) Let K be a KU-subalgebra of H. Let $x, y \in \phi^{-1}(K)$. Then $\phi(x) = a$ and $\phi(y) = b$ for some $a, b \in K$. Thus $\phi(xy) = \phi(x)\phi(y) = ab \in K$, since K is a KU-subalgebra. Hence $xy \in \phi^{-1}(K)$.

(v) Let B be an ideal in $\phi(G)$. Since $\phi(0) = \acute{0}$, $0 \in \phi^{-1}(B)$. Let $x(yz) \in \phi^{-1}(B)$ and $y \in \phi^{-1}(B)$ for $x, y, z \in G$. Then $\phi(x)(\phi(y)\phi(z)) \in B$ and $\phi(y) \in B$. Since B is an ideal of $\phi(G)$, $\phi(x)\phi(z) \in B$. Thus $\phi(xz) \in B$. We get that $xz \in \phi^{-1}B$. Hence $\phi^{-1}B$ is an ideal of G.

(vi) Suppose ϕ is 1-1. Let $x \in \ker\phi$. Then $\phi(x) = \acute{0}$. Since $\phi(0) = \acute{0}$, $\phi(x) = \phi(0)$. Since ϕ is 1-1, $x = 0$. Thus $\ker\phi = \{0\}$.

Conversely, suppose $\ker\phi = \{0\}$. Let $x, y \in G$ be such that $\phi(x) = \phi(y)$. Then we get that

$$\phi(xy) = \phi(x)\phi(y) = \acute{0}$$

and

$$\phi(yx) = \phi(y)\phi(x) = \acute{0}$$

Thus $xy, yx \in \ker\phi$, so $xy = 0 = yx$. It follows that $x = y$. Hence ϕ is 1-1.

Next, we state the first isomorphism of KU-algebras as the following theorem:

Theorem 2. (First Isomorphism Theorem)

If ϕ is an epimorphism from a KU-algebra G onto a KU-algebra H,then the quotient KU-algebra $G/\ker(\phi)$ is isomorphic to H.

Proof of Theorem 2. Let $\phi : G \to H$ be an epimorphism and $\psi : G/A \to H$ defined by $\psi(A_x) = \phi(x)$ for all $A_x \in G/A$, where $A =\ker(\phi)$.

Let $A_x, A_y \in G/A$ be such that $A_x = A_y$. Then $A_{xy} = A_x * A_y = A$ and $A_{yx} = A_y * A_x = A$. So $xy, xy \in A$. Thus $\phi(x)\phi(y) = 0 = \phi(y)\phi(x)$. By (4), we get that $\phi(x) = \phi(y)$. It follows that $\psi(A_x) = \psi(A_y)$. Hence ψ is well-defined.

Let $A_x, A_y \in G/A$ be such that $\psi(A_x) = \psi(A_y)$. Then $\phi(x) = \phi(y)$, so $\phi(x)\phi(y) = \acute{0} = \phi(y)\phi(x)$. Since ϕ is a homomorphism, $\phi(xy) = \acute{0} = \phi(yx)$. Thus $xy, yx \in A$, and we have $x \sim y$. Then we get $A_x = A_y$. Hence ψ is an injection. It is obvious that ψ is a surjection.

Since for all $A_x, A_y \in G/A$

$$\begin{aligned}
\psi(A_x * A_y) &= \psi(A_{xy}) \\
&= \phi(xy) \\
&= \phi(x)\phi(y) \\
&= \psi(A_x)\psi(A_y),
\end{aligned}$$

ψ is a homomorphism. This completes the proof that ψ is an isomorphism. The map ψ is

$$\begin{tikzcd}
G \arrow[r, "\phi"] \arrow[d, "\gamma"'] & H \\
G/\ker\phi \arrow[ur, "\psi"'] &
\end{tikzcd}$$

a canonical map in the sense that if γ is the canonical homomorphism $\gamma : G \to G/\ker(\phi)$ of Theorem 2, then

$$\phi = \psi \circ \gamma.$$

Theorem 3. Let X, Y, Z be KU-algebras. Suppose that $\phi : X \to Y$ is an epimorphism, and let $\psi : X \to Z$ be a homomorphism. If $\ker(\phi) \subseteq \ker(\psi)$, then there exists a unique homomorphism $\gamma : Y \to Z$ such that $\gamma \circ \phi = \psi$.

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[dl, "\gamma"] \\
Z &
\end{tikzcd}$$

Proof of Theorem 3. Let $y \in Y$. Since ϕ is onto, there exists $x_y \in X$ such that $\phi(x_y) = y$. Then we define

$$\gamma(y) = \psi(x_y)$$

To show that γ is well defined, let $a, b \in Y$ be such that $a = b$. Since ϕ is onto, $a = \phi(x_a)$ and $b = \phi(x_b)$ for some $x_a, x_b \in X$. Then $\phi(x_a) = \phi(x_b)$, so

$$\phi(x_a x_b) = \phi(x_a)\phi(x_b) = 0$$

and

$$\phi(x_b x_a) = \phi(x_b)\phi(x_a) = 0.$$

Then $x_a x_b, x_b x_a \in \ker(\phi)$. Since $\ker(\phi) \subseteq \ker(\psi)$, $\psi(x_a x_b) = 0$ and $\psi(x_b x_a) = 0$. Then

$$\psi(x_a x_b) = 0 = \psi(x_b x_a).$$

By (4), we get that $\psi(x_a) = \psi(x_b)$. That is $\gamma(a) = \gamma(b)$. Hence γ is well defined.

To show that $\gamma \circ \phi = \psi$, let $x \in X$. Then $\phi(x) = y$ for some $y \in Y$. Now we have $\gamma(y) = \psi(x)$. Thus

$$\begin{aligned}(\gamma \circ \phi)(x) &= \gamma(\phi(x)) \\ &= \gamma(y) \\ &= \psi(x).\end{aligned}$$

Hence $\gamma \circ \phi = \psi$.

Next, we show that γ is homomorphism. Let $a, b \in Y$. Then there exist $x_a, x_b \in X$ such that $a = \phi(x_a)$ and $b = \phi(x_b)$. Thus $\gamma(a) = \psi(x_a)$ and $\gamma(b) = \psi(x_b)$. Now we have

$$ab = \phi(x_a)\phi(x_b) = \phi(x_a x_b).$$

The equation

$$\begin{aligned}\gamma(ab) &= \psi(x_a x_b) \\ &= \psi(x_a)\psi(x_b) \\ &= \gamma(a)\gamma(b)\end{aligned}$$

shows that γ is a homomorphism.

Finally, if $\beta : Y \to Z$ is another function such that $\beta \circ \phi = \psi$. Then for all $y \in Y$ there exists $x_y \in X$ such that $y = \phi(x_y)$. Thus

$$\begin{aligned}\gamma(y) &= \psi(x_y) \\ &= (\beta \circ \phi)(x_y) \\ &= \beta(\phi(x_y)) \\ &= \beta(y).\end{aligned}$$

This completes the proof.

Corollary 1. Let X, Y, be KU-algebras, let A be an ideal of X, and let ϕ be a canonical mapping from X onto X/A. If ψ is a homomorphism from X to Y and $A \subseteq \ker(\psi)$, then there exists a unique homomorphism $\gamma : X/A \to Y$ such that $\gamma \circ \phi = \psi$.

Theorem 4. (Second Isomorphism Theorem)

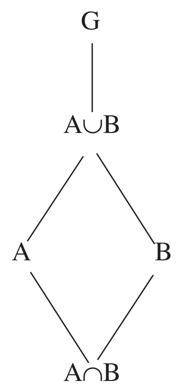
Let G be a KU-algebra. Let A, B be ideals of G. If $A \cup B$ is a KU-algebra, then the quotient KU-algebras $(A \cup B)/B$ and $A/(A \cap B)$ are isomorphic.

Proof of Theorem 4. Let $\phi : A \to (A \cup B)/B$ defined by $\phi(x) = B_x$ for all $x \in A$. It is obvious that ϕ is well defined. Let $B_x \in (A \cup B)/B$. If $x \in A$, then $B_x = \phi(x)$. If x$\in B$, then $B_x = B_0 = \phi(0)$. Thus ϕ is onto $(A \cup B)/B$. The equation

$$\begin{aligned}\phi(xy) &= B_{xy} \\ &= B_x * B_y \\ &= \phi(x)\phi(y)\end{aligned}$$

shows that ϕ is a homomorphism.
Now let $x \in \ker\phi$. Then we get $\phi(x) = B_0$, so $B_x = B_0$. Thus $x \in B$. Since $\ker\phi \subseteq A$, $x \in A \cap B$. Hence $\ker\phi \subseteq A \cap B$. On the other hand, let $x \in A \cap B$. Then $x \in B$. Thus $\phi(x) = B_x = B_0$, so $x \in \ker\phi$. Hence $A \cap B \subseteq \ker\phi$. Therefore, $\ker\phi = A \cap B$. Theorem 2. immediately gives us that $(A \cup B)/B \cong A/(A \cap B)$.

We can describe Theorem 4 by the following figure:



Next, we state the third isomorphism theorem of KU-algebras.

Theorem 5. (Third Isomorphism Theorem)

Let G be a KU-algebra. Let A and B be ideals of G, with $A \subseteq B \subseteq G$. Then

(i) the quotient B/A is an ideal of the quotient G/A, and

(ii) the quotient KU-algebra $(G/A)/(B/A)$ is isomorphic to G/B.

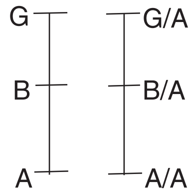
Proof of Theorem 5.

(i) To show that B/A is an ideal of G/A, let $A_x * (A_y * A_z) \in B/A$ and $A_y \in B/A$. Then $x(yz) \in B$ and $y \in B$. Since B is an ideal of G, $xy \in B$, so $A_{xz} \in B/A$. Thus $A_x * A_z \in B/A$. It is clear that $A \in B/A$. Therefore, B/A is an ideal of G/A.

(ii) Let $\phi : G/A \to G/B$ defined by $\phi(A_x) = B_x$. Let $A_x = A_y$. Then $x \sim y$ determined by A, that is $xy, yx \in A$. Since $A \subseteq B$, $xy, yx \in B$. Thus $x \sim y$ determined by B, and hence $B_x = B_y$. Then $\phi(A_x) = \phi(A_y)$. Therefore, ϕ is well defined. To show that ϕ is onto G/B, let $B_x \in G/B$. If $x \in G$ and $x \notin B$, then $B_x = \phi(A_x)$. If $x \in B$, then $B_x = B_0 = \phi(B_0)$. Hence ϕ is onto. The equation

$$\begin{aligned} \phi(A_x * A_y) &= \phi(A_{xy}) \\ &= B_{xy} \\ &= B_x * B_y \\ &= \phi(A_x) * \phi(A_y) \end{aligned}$$

shows that ϕ is a homomorphism.
To show that $\ker\phi = B/A$, let $A_x \in \ker\phi$. Then $\phi(A_x) = B_0$, so $B_x = B_0$. Thus $x \in B$. Now we have $A_x \in B/A$. Hence $\ker\phi \subseteq B/A$. Going the other hand, let $A_x \in B/A$. $\phi(A_x) = B_x = B_0$, since $x \in B$. Thus $A_x \in \ker\phi$, and hence $A/B \subseteq \ker\phi$. Therefore, $\ker\phi = B/A$. By Theorem 2, $(G/A)/(B/A)$ is isomorphic to G/B.



It turns out that an analogous result of the third isomorphism theorem for groups is also true for KU-algebras.

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Smarandache friendly numbers-another approach

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Abstract One approach to Smarandache friendly numbers is given by A.Murthy, who defined them Ref [1]. Another approach is presented here.

Keywords Smarandache friendly numbers.

Smarandache friendly numbers were defined by A. Murthy [1] as follows.

Definition 1. If the sum of any set of consecutive terms of a sequence is equal to the product of first and last number, then the first and the last numbers are called a pair of Smarandache friendly numbers.

Here, we will consider a sequence of natural numbers.

1. It is easy to note that $(3, 6)$ is a friendly pair as $3 + 4 + 5 + 6 = 18 = 3 \cdot 6$.

2. By elementary operations and trial, we can find such pairs, but as magnitude of natural numbers increases, this work becomes tedious. Hence an algorithm is presented here.

Assume that (m, n) is a pair of friendly numbers, $n > m$, so that

$$m + (m + 1) + (m + 2) + \cdots + n = m \cdot n.$$

Let $n = m + k$, where k is a natural number. Then the above equation becomes

$$m + (m + 1) + (m + 2) + \cdots + n = m \cdot (m + k).$$

On simplification, this gives,

$$k^2 + k - 2(m^2 - m) = 0.$$

That is,

$$k = \frac{-1 + \sqrt{1 + 8(m^2 - m)}}{2},$$

considering the positive sign only.

Now, k will be a natural number only if $1+8(m^2 - m)$ is a perfect square of an odd natural number.

For $m = 3$, we have $k = 3$, so that $n = 3 + 3 = 6$, and then $(3, 6)$ are friendly numbers as we observed earlier.

For $m = 5, k = \frac{-1+\sqrt{161}}{2}$, which is not an integer. Hence k does not exist for every m.

For $m = 15, k = 20$, hence $n = 35$. So the next pair of friendly numbers is $(15, 35)$. Other pairs are $(85, 204)$ and $(493, 1189)$.

At the end, the list of m and $1 + 8(m^2 - m)$ is given using a computer software.

2. If (m, n) is a friendly pair, of natural numbers, then it is conjectured that $(m+2n, 2m+5n-1)$ is also a friendly pair.

Since, $(3, 6)$ and $(15, 35)$ are frendly pairs, we have $3x + 6y = 15$ and $3p + 6q = 35$, for some x, y, p, q being natural numbers. These equations are true for $x = 1, y = 2, p = 2 \text{and} q = 5$. With 1 to be substracted from second equation.

These solutions are unique, hence this conjecture.

This suggests that there are infinite pairs of friendly natural numbers.

Definition 2. [1] x and y are primes. They are called Smarandache friendly primes if the sum of any set of consecutive primes, whose first term is x and last is y, is equal to their product $x \cdot y$.

Example. $(2, 5), (3, 13), (5, 31)$ are pairs of friendly primes, for $3 + 5 + 7 + 11 + 13 = 39 = 3 \cdot 13$.

Since the primes are not uniformly distributed, an algorithm for friendly primes seems to be impossible.

4. For various sequences, different pairs of friendly numbers can be obtained.

The values of $m \leq 1000$ and $1 + 8(m^2 - m)$ are given below. Those with perfect squares are underlined. Only four pairs were obtained.

1	1.0000	2	4.1231	**<u>3</u>**	**<u>7.0000</u>**	4	9.8489	5	12.6886
6	15.5242	7	18.3576	8	21.1896	9	24.0208	10	26.8514
11	29.6816	12	32.5115	13	35.3412	14	38.1707	**<u>15</u>**	**<u>41.0000</u>**
16	43.8292	17	46.6583	18	49.4874	19	52.3163	20	55.1453
21	57.9741	22	60.8030	23	63.6318	24	66.4605	25	69.2892
26	72.1180	17	74.9466	28	77.7753	29	80.6040	30	83.4326
31	86.2612	32	89.0898	33	91.9184	34	94.7470	35	97.5756
36	100.4042	37	103.2327	38	106.0613	39	108.8899	40	111.7184
41	114.5469	42	117.3755	43	120.2040	44	123.0325	45	125.8610
46	128.6895	47	131.5181	48	134.3466	49	137.1751	50	140.0036
51	142.8321	52	145.6606	53	148.4891	54	151.3176	55	154.1460
56	156.9745	57	159.8030	58	162.6315	59	165.4600	60	168.2884
61	171.1169	62	173.9454	63	176.7739	64	179.6023	65	182.4308
66	185.2593	67	188.0877	68	190.9162	69	193.7447	70	196.5731

71	199.4016	72	202.2301	73	205.0585	74	207.8870	75	210.7155
76	213.5439	77	216.3724	78	219.2008	79	222.0293	80	224.8577
81	227.6862	82	230.5146	83	233.3431	84	236.1716	**85**	**239.0000**
86	241.8284	87	244.6569	88	247.4854	89	250.3138	90	253.1423
91	255.9707	92	258.7992	93	261.6276	94	264.4561	95	267.2845
96	270.1129	97	272.9414	98	275.7698	99	278.5983	100	281.4267
101	284.2552	102	287.0836	103	289.9120	104	292.7405	105	295.5689
106	298.3974	107	301.2258	108	304.0543	109	306.8827	110	309.7112
111	312.5396	112	315.3680	113	318.1965	114	321.0249	115	323.8534
116	326.6818	117	329.5103	118	332.3387	119	335.1671	120	337.9956
121	340.8240	122	343.6524	123	346.4809	124	349.3093	125	352.1378
126	354.9662	127	357.7946	128	360.6231	129	363.4515	130	366.2799
131	369.1084	132	371.9368	133	374.7653	134	377.5937	135	380.4221
136	383.2506	137	386.0790	138	388.9074	139	391.7359	140	394.5643
141	397.3928	142	400.2212	143	403.0496	144	405.8781	145	408.7065
146	411.5349	147	414.3634	148	417.1918	149	420.0202	150	422.8487
151	425.6771	152	428.5056	153	431.3340	154	434.1624	155	436.9908
156	439.8193	157	442.6477	158	445.4761	159	448.3046	160	451.1330
161	453.9615	162	456.7899	163	459.6183	164	462.4467	165	465.2752
166	468.1036	167	470.9321	168	473.7605	169	476.5889	170	479.4174
171	482.2458	172	485.0742	173	487.9026	174	490.7311	175	493.5595
176	496.3879	177	499.2164	178	502.0448	179	504.8733	180	507.7017
181	510.5301	182	513.3585	183	516.1870	184	519.0154	185	521.8439
186	524.6723	187	527.5007	188	530.3292	189	533.1576	190	535.9860
191	538.8145	192	541.6429	193	544.4713	194	547.2997	195	550.1282
196	552.9566	197	555.7850	198	558.6135	199	561.4419	200	564.2703
201	567.0988	202	569.9272	203	572.7556	204	575.5840	205	578.4125
206	581.2409	207	584.0693	208	586.8978	209	589.7262	210	592.5546
211	595.3831	212	598.2115	213	601.0399	214	603.8683	215	606.6968
216	609.5252	217	612.3536	218	615.1821	219	618.0105	220	620.8389
221	623.6674	222	626.4958	223	629.3242	224	632.1526	225	634.9811
226	637.8095	227	640.6379	228	643.4664	229	646.2948	230	649.1232
231	651.9517	232	654.7801	233	657.6085	234	660.4370	235	663.2654
236	666.0938	237	668.9222	238	671.7507	239	674.5791	240	677.4075
241	680.2360	242	683.0644	243	685.8928	244	688.7213	245	691.5497
246	694.3781	247	697.2065	248	700.0350	249	702.8634	250	705.6918
251	708.5203	252	711.3487	253	714.1771	254	717.0056	255	719.8340
256	722.6624	257	725.4908	258	728.3193	259	731.1477	260	733.9761
261	736.8046	262	739.6330	263	742.4614	264	745.2899	265	748.1183
266	750.9467	267	753.7751	268	756.6036	269	759.4320	270	762.2604

271	765.0889	272	767.9173	273	770.7457	274	73.5742
275	776.4026	276	779.2310	277	782.0594	278	784.8879
279	787.7163	280	790.5447	281	793.3732	282	796.2016
283	799.0300	284	801.8585	285	804.6869	286	807.5153
287	810.3438	288	813.1722	289	816.0006	290	818.8290
291	821.6575	292	824.4859	293	827.3143	294	830.1428
295	832.9712	296	835.7996	297	838.6281	298	841.4565
299	844.2849	300	847.1133	301	849.9418	302	852.7702
303	855.5986	304	858.4271	305	861.2555	306	864.0839
307	866.9124	308	869.7408	309	872.5692	310	875.3976
311	878.2261	312	881.0545	313	883.8829	314	886.7114
315	889.5398	316	892.3682	317	895.1967	318	898.0251
319	900.8535	320	903.6819	321	906.5103	322	909.3387
323	912.1672	324	914.9956	325	917.8240	326	920.6525
327	923.4809	328	926.3093	329	929.1378	330	931.9662
331	934.7946	332	937.6230	333	940.4515	334	943.2799
335	946.1083	336	948.9368	337	951.7652	338	954.5936
339	957.4221	340	960.2505	341	963.0789	342	965.9073
343	968.7358	344	971.5642	345	974.3926	346	977.2211
347	980.0495	348	982.8779	349	985.7064	350	988.5348
351	991.3632	352	994.1917	353	997.0201	354	999.8485
355	1002.6769	356	1005.5054	357	1008.3338	358	1011.1622
359	1013.9907	360	1016.8190	361	1019.6475	362	1022.4759
363	1025.3043	364	1028.1328	365	1030.9612	366	1033.7897
367	1036.6180	368	1039.4465	369	1042.2749	370	1045.1034
371	1047.9318	372	1050.7603	373	1053.5886	374	1056.4171
375	1059.2455	376	1062.0740	377	1064.9023	378	1067.7307
379	1070.5592	380	1073.3876	381	1076.2161	382	1079.0444
383	1081.8729	384	1084.7013	385	1087.5298	386	1090.3582
387	1093.1866	388	1096.0150	389	1098.8435	390	1101.6719
391	1104.5004	392	1107.3287	393	1110.1572	394	1112.9856
395	1115.8141	396	1118.6425	397	1121.4709	398	1124.2993
399	1127.1278	400	1129.9562	401	1132.7847	402	1135.6130
403	1138.4415	404	1141.2699	405	1144.0984	406	1146.9268
407	1149.7552	408	1152.5836	409	1155.4120	410	1158.2405
411	1161.0688	412	1163.8973	413	1166.7257	414	1169.5542
415	1172.3826	416	1175.2111	417	1178.0394	418	1180.8679
419	1183.6963	420	1186.5248	421	1189.3531	422	1192.1816
423	1195.0100	424	1197.8385	425	1200.6669	426	1203.4954
427	1206.3237	428	1209.1522	429	1211.9806	430	1214.8091

431	1217.6375	432	1220.4659	433	1223.2943	434	1226.1228
435	1228.9512	436	1231.7797	437	1234.6080	438	1237.4365
439	1240.2649	440	1243.0933	441	1245.9218	442	1248.7501
443	1251.5786	444	1254.4070	445	1257.2355	446	1260.0638
447	1262.8923	448	1265.7207	449	1268.5492	450	1271.3776
451	1274.2061	452	1277.0344	453	1279.8629	454	1282.6913
455	1285.5198	456	1288.3481	457	1291.1766	458	1294.0050
459	1296.8335	460	1299.6619	461	1302.4904	462	1305.3187
463	1308.1472	464	1310.9756	465	1313.8041	466	1316.6324
467	1319.4608	468	1322.2893	469	1325.1177	470	1327.9462
471	1330.7745	472	1333.6030	473	1336.4314	474	1339.2599
475	1342.0883	476	1344.9167	477	1347.7451	478	1350.5736
479	1353.4020	480	1356.2305	481	1359.0588	482	1361.8873
483	1364.7157	484	1367.5442	485	1370.3726	486	1373.2010
487	1376.0294	488	1378.8579	489	1381.6863	490	1384.5148
491	1387.3431	492	1390.1716	**493**	**1393.0000**	494	1395.8284
495	1398.6569	496	1401.4852	497	1404.3137	498	1407.1421
499	1409.9706	500	1412.7990	501	1415.6274	502	1418.4558
503	1421.2843	504	1424.1127	505	1426.9412	506	1429.7695
507	1432.5980	508	1435.4264	509	1438.2549	510	1441.0833
511	1443.9117	512	1446.7401	513	1449.5686	514	1452.3970
515	1455.2255	516	1458.0538	517	1460.8823	518	1463.7107
519	1466.5391	520	1469.3676	521	1472.1959	522	1475.0244
523	1477.8528	524	1480.6813	525	1483.5096	526	1486.3381
527	1489.1665	528	1491.9950	529	1494.8234	530	1497.6519
531	1500.4802	532	1503.3087	533	1506.1371	534	1508.9656
535	1511.7939	536	1514.6224	537	1517.4508	538	1520.2793
539	1523.1077	540	1525.9362	541	1528.7645	542	1531.5930
543	1534.4214	544	1537.2499	545	1540.0782	546	1542.9066
547	1545.7351	548	1548.5635	549	1551.3920	550	1554.2203
551	1557.0488	552	1559.8772	553	1562.7057	554	1565.5341
555	1568.3625	556	1571.1909	557	1574.0194	558	1576.8478
559	1579.6763	560	1582.5046	561	1585.3331	562	1588.1615
563	1590.9900	564	1593.8184	565	1596.6469	566	1599.4752
567	1602.3037	568	1605.1321	569	1607.9604	570	1610.7889
571	1613.6173	572	1616.4458	573	1619.2742	574	1622.1027
575	1624.9310	576	1627.7595	577	1630.5879	578	1633.4164
579	1636.2448	580	1639.0732	581	1641.9016	582	1644.7301
583	1647.5585	584	1650.3870	585	1653.2153	586	1656.0438
587	1658.8722	588	1661.7007	589	1664.5291	590	1667.3575

591	1670.1859	592	1673.0144	593	1675.8428	594	1678.6711
595	1681.4996	596	1684.3280	597	1687.1565	598	1689.9849
599	1692.8134	600	1695.6417	601	1698.4702	602	1701.2986
603	1704.1271	604	1706.9554	605	1709.7839	606	1712.6123
607	1715.4408	608	1718.2692	609	1721.0977	610	1723.9260
611	1726.7545	612	1729.5829	613	1732.4114	614	1735.2397
615	1738.0682	616	1740.8966	617	1743.7250	618	1746.5535
619	1749.3818	620	1752.2103	621	1755.0387	622	1757.8672
623	1760.6956	624	1763.5240	625	1766.3524	626	1769.1809
627	1772.0093	628	1774.8378	629	1777.6661	630	1780.4946
631	1783.3230	632	1786.1515	633	1788.9799	634	1791.8083
635	1794.6367	636	1797.4652	637	1800.2936	638	1803.1221
639	1805.9504	640	1808.7789	641	1811.6073	642	1814.4357
643	1817.2642	644	1820.0925	645	1822.9210	646	1825.7494
647	1828.5779	648	1831.4063	649	1834.2347	650	1837.0631
651	1839.8916	652	1842.7200	653	1845.5485	654	1848.3768
655	1851.2053	656	1854.0337	657	1856.8622	658	1859.6906
659	1862.5190	660	1865.3474	661	1868.1759	662	1871.0043
663	1873.8328	664	1876.6611	665	1879.4895	666	1882.3180
667	1885.1464	668	1887.9749	669	1890.8032	670	1893.6317
671	1896.4601	672	1899.2886	673	1902.1169	674	1904.9454
675	1907.7738	676	1910.6023	677	1913.4307	678	1916.2592
679	1919.0875	680	1921.9160	681	1924.7444	682	1927.5729
683	1930.4012	684	1933.2297	685	1936.0581	686	1938.8866
687	1941.7150	688	1944.5433	689	1947.3718	690	1950.2002
691	1953.0287	692	1955.8571	693	1958.6855	694	1961.5139
695	1964.3424	696	1967.1708	697	1969.9993	698	1972.8276
699	1975.6561	700	1978.4845	701	1981.3130	702	1984.1414
703	1986.9698	704	1989.7982	705	1992.6267	706	1995.4551
707	1998.2836	708	2001.1119	709	2003.9403	710	2006.7688
711	2009.5972	712	2012.4257	713	2015.2540	714	2018.0825
715	2020.9109	716	2023.7394	717	2026.5677	718	2029.3962
719	2032.2246	720	2035.0531	721	2037.8815	722	2040.7100
723	2043.5383	724	2046.3668	725	2049.1953	726	2052.0237
727	2054.8521	728	2057.6804	729	2060.5090	730	2063.3374
731	2066.1658	732	2068.9941	733	2071.8225	734	2074.6511
735	2077.4795	736	2080.3079	737	2083.1362	738	2085.9648
739	2088.7932	740	2091.6216	741	2094.4500	742	2097.2786
743	2100.1069	744	2102.9353	745	2105.7637	746	2108.5923
747	2111.4207	748	2114.2490	749	2117.0774	750	2119.9060

751	2122.7344	752	2125.5627	753	2128.3911	754	2131.2195
755	2134.0481	756	2136.8765	757	2139.7048	758	2142.5332
759	2145.3618	760	2148.1902	761	2151.0186	762	2153.8469
763	2156.6755	764	2159.5039	765	2162.3323	766	2165.1606
767	2167.9893	768	2170.8176	769	2173.6460	770	2176.4744
771	2179.3030	772	2182.1313	773	2184.9597	774	2187.7881
775	2190.6167	776	2193.4451	777	2196.2734	778	2199.1018
779	2201.9302	780	2204.7588	781	2207.5872	782	2210.4155
783	2213.2439	784	2216.0725	785	2218.9009	786	2221.7292
787	2224.5576	788	2227.3862	789	2230.2146	790	2233.0430
791	2235.8713	792	2238.7000	793	2241.5283	794	2244.3567
795	2247.1851	796	2250.0137	797	2252.8420	798	2255.6704
799	2258.4988	800	2261.3271	801	2264.1558	802	2266.9841
803	2269.8125	804	2272.6409	805	2275.4695	806	2278.2979
807	2281.1262	808	2283.9546	809	2286.7832	810	2289.6116
811	2292.4399	812	2295.2683	813	2298.0969	814	2300.9253
815	2303.7537	816	2306.5820	817	2309.4106	818	2312.2390
819	2315.0674	820	2317.8958	821	2320.7241	822	2323.5527
823	2326.3811	824	2329.2095	825	2332.0378	826	2334.8665
827	2337.6948	828	2340.5232	829	2343.3516	830	2346.1802
831	2349.0085	832	2351.8369	833	2354.6653	834	2357.4939
835	2360.3223	836	2363.1506	837	2365.9790	838	2368.8076
839	2371.6360	840	2374.4644	841	2377.2927	842	2380.1211
843	2382.9497	844	2385.7781	845	2388.6064	846	2391.4348
847	2394.2634	848	2397.0918	849	2399.9202	850	2402.7485
851	2405.5771	852	2408.4055	853	2411.2339	854	2414.0623
855	2416.8909	856	2419.7192	857	2422.5476	858	2425.3760
859	2428.2046	860	2431.0330	861	2433.8613	862	2436.6897
863	2439.5183	864	2442.3467	865	2445.1750	866	2448.0034
867	2450.8318	868	2453.6604	869	2456.4888	870	2459.3171
871	2462.1455	872	2464.9741	873	2467.8025	874	2470.6309
875	2473.4592	876	2476.2878	877	2479.1162	878	2481.9446
879	2484.7729	880	2487.6016	881	2490.4299	882	2493.2583
883	2496.0867	884	2498.9153	885	2501.7437	886	2504.5720
887	2507.4004	888	2510.2288	889	2513.0574	890	2515.8857
891	2518.7141	892	2521.5425	893	2524.3711	894	2527.1995
895	2530.0278	896	2532.8562	897	2535.6848	898	2538.5132
899	2541.3416	900	2544.1699	901	2546.9985	902	2549.8269
903	2552.6553	904	2555.4836	905	2558.3123	906	2561.1406
907	2563.9690	908	2566.7974	909	2569.6257	910	2572.4543

911	2575.2827	912	2578.1111	913	2580.9395	914	2583.7681
915	2586.5964	916	2589.4248	917	2592.2532	918	2595.0818
919	2597.9102	920	2600.7385	921	2603.5669	922	2606.3955
923	2609.2239	924	2612.0522	925	2614.8806	926	2617.7092
927	2620.5376	928	2623.3660	929	2626.1943	930	2629.0227
931	2631.8513	932	2634.6797	933	2637.5081	934	2640.3364
935	2643.1650	936	2645.9934	937	2648.8218	938	2651.6501
939	2654.4788	940	2657.3071	941	2660.1355	942	2662.9639
943	2665.7925	944	2668.6208	945	2671.4492	946	2674.2776
947	2677.1062	948	2679.9346	949	2682.7629	950	2685.5913
951	2688.4197	952	2691.2483	953	2694.0767	954	2696.9050
955	2699.7334	956	2702.5620	957	2705.3904	958	2708.2188
959	2711.0471	960	2713.8757	961	2716.7041	962	2719.5325
963	2722.3608	964	2725.1895	965	2728.0178	966	2730.8462
967	2733.6746	968	2736.5032	969	2739.3315	970	2742.1599
971	2744.9883	972	2747.8167	973	2750.6453	974	2753.4736
975	2756.3020	976	2759.1304	977	2761.9590	978	2764.7874
979	2767.6157	980	2770.4441	981	2773.2727	982	2776.1011
983	2778.9294	984	2781.7578	985	2784.5864	986	2787.4148
987	2790.2432	988	2793.0715	989	2795.9001	990	2798.7285
991	2801.5569	992	2804.3853	993	2807.2136	994	2810.0422
995	2812.8706	996	2815.6990	997	2818.5273	998	2821.3560
999	2824.1843	1000	2827.0127				

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Simple closed null curves in Minkowski 3-Space

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Abstract In this work, we present some characterizations of simple closed null curves in Minkowski 3-space. Moreover, in the same space, we study a special case of such curves "Null curves of constant breadth".

Keywords Minkowski 3-Space, simple closed null curve, curves of constant breadth.

§1. Introduction

At the beginning of the twentieth century, A. Einstein's theory opened a door to new geometries such as Minkowski space, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold.

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifolds. For instance, in [6] and [10], the authors studied space-like and time-like curves of constant breadth in Minkowski 3-space, respectively. These studies have been treated based of the papers [1], [3], [8] and [9], in the spaces E^2, E^3 and E^4.

In this work, we investigate position vector of a simple closed null curve and give some characterizations in Minkowski space E_1^3. Additionally, we express a characterization in the case of constant breadth.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^3 are briefly presented (A more complete elementary treatment can be found in [2]).

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since g is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null if $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\varphi = \varphi(s)$ in E_1^3 can locally be space-like, time-like or null (light-like), if all of its velocity vectors φ' are respectively space-like, time-like or null (light-like), for every $s \in I \subset R$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|g(a, a)|}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $g(v, w) = 0$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve φ in the space E_1^3. For an arbitrary curve φ with first and second curvature, κ and τ in the space E_1^3, the following Frenet formulae are given in [4]:

If φ is a null curve, then the Frenet formulae has the form

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \tau & 0 & -\kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

satisfying equations

$$g(T, T) = g(B, B) = g(T, N) = g(N, B) = 0,$$
$$g(N, N) = 1,\ g(T, B) = 1.$$

In this case, κ can take only two values: $\kappa = 0$ when α is a null straight line, or $\kappa = 1$ in all other cases.

Recall that an arbitrary curve is called a $W-$curve if it has constant Frenet curvatures [7]. In the rest of the paper, we shall assume $\kappa = 1$ at every point.

§3. Simple closed null curves in Minkowski 3-Space

Let $\varphi = \varphi(s)$ be null curve in the space E_1^3. Moreover, let us suppose $\varphi = \varphi(s)$ simple closed null curve in the space E_1^3. These curves will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other than P. We call the point Q the opposite point of P. We consider a curve in the class Γ as in [5] having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve. Here, we suppose both φ and φ^* are unit speed null curves of E_1^3. A simple closed null curve having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\varphi^* = \varphi + \lambda T + \mu N + \delta B. \tag{2}$$

where λ, μ and δ are arbitrary functions of s and φ and φ^* are opposite points. Differentiating both sides of (2) and considering Frenet equations, we have

$$\frac{d\varphi^*}{ds} = T^* = \left\{\frac{d\lambda}{ds} + \mu\tau + 1\right\} T + \left\{\frac{d\mu}{ds} + \lambda - \delta\tau\right\} N + \left\{\frac{d\delta}{ds} - \mu\right\} B. \tag{3}$$

We know that $T^* = -T$. Then, we get the following system of ordinary differential equations:

$$\begin{aligned} \lambda' &= -\mu\tau - 2 \\ \mu' &= \delta\tau - \lambda \\ \delta' &= \mu \end{aligned} \quad . \tag{4}$$

Using system (4), we have the following differential equation with respect to λ as

$$\frac{d}{ds}\left\{\frac{1}{\tau}\frac{d}{ds}\left[\frac{1}{\tau}\left(\frac{d\lambda}{ds} + 2\right)\right]\right\} - \frac{d}{ds}\left(\frac{\lambda}{\tau}\right) - \frac{1}{\tau}\left(\frac{d\lambda}{ds} + 2\right) = 0. \tag{5}$$

Theorem 3.1. This obtained differential equation of third order with variable coefficients (5) is a characterization for the simple closed null curve in E_1^3. Via its solution, position vector of φ^* can be determined.

However, a general solution of (5) has not yet been found. Let us suppose $\varphi = \varphi(s)$ be a null $W-$curve in the space E_1^3. In this case the differential equation (5) transforms to

$$\frac{d^3\lambda}{ds^3} - 2\tau\frac{d\lambda}{ds} - 2\tau = 0. \tag{6}$$

According to signature of τ, we study the following cases.

Case 3.1. $\tau > 0$. Then, we have the solution of the differential equation (6) as

$$\lambda = \Phi_1 \cosh\sqrt{2\tau}s + \Phi_2 \sinh\sqrt{2\tau}s - s \tag{7}$$

where $\Phi_1, \Phi_2 \in R$. Thus, we may express other components, respectively,

$$\mu = -\sqrt{\frac{2}{\tau}}\left(\Phi_1 \sinh\sqrt{2\tau}s + \Phi_2 \cosh\sqrt{2\tau}s + \frac{1}{\sqrt{2\tau}}\right) \tag{8}$$

and

$$\delta = -\frac{1}{\tau}\left(\Phi_1 \cosh\sqrt{2\tau}s + \Phi_2 \sinh\sqrt{2\tau}s - s\right). \tag{9}$$

Case 3.2. $\tau < 0$. In this case, we have the solution

$$\lambda = \Psi_1 \cos\sqrt{-2\tau}s + \Psi_2 \sin\sqrt{-2\tau}s - s \tag{10}$$

for the real numbers $\Psi_1, \Psi_2 \in R$. Since, we write other components

$$\mu = -\frac{1}{\tau}\sqrt{-2\tau}\left(-\Phi_1 \sin\sqrt{-2\tau}s + \Phi_2 \cos\sqrt{-2\tau}s + \frac{1}{\sqrt{-2\tau}}\right) \tag{11}$$

and

$$\delta = -\frac{1}{\tau}\left(\Psi_1 \cos\sqrt{-2\tau}s + \Psi_2 \sin\sqrt{-2\tau}s - s\right). \tag{12}$$

Theorem 3.2. Position vector of a simple closed null $W-$curve in Minkowski space E_1^3 can be composed by the components (7), (8) and (9); or (10), (11) and (12) according to singature of the torsion.

§4. Null curves of constant breadth in Minkowski 3-Space

In this section, we give a characterization of the null curves of constant breadth in Minkowski 3-space.

Let us suppose the null curve treated in the previous section has constant breadth. In another words, let the distance between opposite points of C and C^* be constant. Then, due to null frame vectors, we may express

$$\|\varphi^* - \varphi\| = 2\lambda\delta + \mu^2 = l^2 = \text{constant}. \tag{13}$$

Differentiating both sides of (13), we immediately arrive

$$\delta\frac{d\lambda}{ds} + \lambda\frac{d\delta}{ds} + \mu\frac{d\mu}{ds} = 0. \tag{14}$$

Considering system of ordinary differential equations (4), we have

$$\delta = 0 \tag{15}$$

and, thereafter

$$\lambda = \mu = 0. \tag{16}$$

Since, we give:

Theorem 4.1. There does not exist null curves of constant breadth in Minkowski 3-space E_1^3.

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On the Elliptic cylindrical Tzitzeica curves in Minkowski 3-Space

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Abstract Elliptic cylindrical curves satisfying Tzitzeica condition are obtained via the solution of the forced harmonic equation in Minkowski 3-Space. In addition, we have given the conditions to be of spacelike, timelike and null curve of the elliptic cylindrical Tzitzeica curve.

Keywords Tzitzeica curve, Elliptic cylindrical curve, Minkowski 3-Space.

§1. Preliminaries and Introduction

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the Lorentzian inner product

$$\langle x, y\rangle_L = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. An arbitrary vector $x = (x_1, x_2, x_3)$ in E_1^3can have one of three Lorentzian causal characters: it is spacelike if $\langle x, x\rangle_L > 0$ or $x = 0$, timelike if $\langle x, x\rangle_L < 0$ and null (lightlike) if $\langle x, x\rangle_L = 0$ and $x \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 is locally spacelike, timelike or null (lightlike), if all of its velocity vectors (tangents) $\alpha'(s) = T(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset IR$. Lorentzian vectoral product of x and y is defined by

$$x \wedge_L y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2).$$

Recall that the pseudo-norm of an arbitrary vector $x \in E_1^3$ is given by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$. If the curve α is non-unit speed, then

$$\kappa(t) = \frac{\left\|\alpha^{'}(t) \wedge_L \alpha^{''}(t)\right\|_L}{\left\|\alpha^{'}(t)\right\|_L^3}, \tau(t) = \frac{\det\left(\alpha^{'}(t), \alpha^{''}(t), \alpha^{'''}(t)\right)}{\left\|\alpha^{'}(t) \wedge_L \alpha^{''}(t)\right\|_L^2}. \tag{1.1}$$

If the curve α is unit speed, then

$$\kappa(s) = \|\alpha''(s)\|_L,\ \tau(s) = \|B'(s)\|_L. \tag{1.2}$$

[1,4].

In this paper we have interested in Tzitzeica elliptic cylindrical curves in Minkowski 3-Space, more precisely we ask in what conditions a cylindrical curve is a Tzitzeica one, namely

the function $t \to \frac{\tau(t)}{d^2(t)}$ is constant, where $d(t)$ is the distance from origin to the osculating plane of curve. The Tzitzeica condition yields a third-order ODE which in our framework admits a direct integration. Therefore the final answer of main problem is given via a second order ODE which in the elliptic case is exactly the equation of a forced harmonic oscillator. In this case, the solution depends of four real constants: one defining the Tzitzeica condition and other three obtained by integration.

§2. Elliptic Cylindrical Tzitzeica curves in Minkowski 3-Space

Proposition 1. Let $\alpha(t)$ be an elliptic cylindrical curve in Minkowski 3-Space.Then, the curve $\alpha(t)$ is Tzitzeica curve if and only if

$$f(t) = f(0)\cos t + f'(0)\sin t + \int_0^t \frac{\sin(u-t)}{Ku+c}du,$$

where $f(0)$, $f'(0)$, $K \neq 0$ and c are real constants.

Proof. Let in E_1^3 a curve C given in vectorial form $C : \alpha = \alpha(t)$. This curve is called elliptic cylindrical if has the expression

$$\alpha(t) = (\cos t, \sin t, f(t)) \tag{2.1}$$

and differentiation of $\alpha(t)$, we have

$$\begin{aligned} \alpha'(t) &= (-\sin t, \cos t, f'(t)) \\ \alpha''(t) &= (-\cos t, -\sin t, f''(t)) \\ \alpha'''(t) &= (\sin t, -\cos t, f''(t)) \end{aligned} \tag{2.2}$$

for some $f \in C^\infty(R)$. From Eq.(1.1) and Eq.(2.2), the torsion function is

$$\tau(t) = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge_L \alpha''\|_L^2} = \frac{f' + f'''}{f'^2 + f''^2 - 1}.$$

Then the distance from origin to the osculating plane is

$$d(t) = \frac{|f + f''|}{\sqrt{|f'^2 + f''^2 - 1|}}.$$

Let us suppose that the curve is Tzitzeica with the constant $K \neq 0$, because the curve is not contained in a plane

$$\begin{aligned} K(t) &= \frac{\tau(t)}{d^2(t)} \\ &= \frac{f' + f'''}{f'^2 + f''^2 - 1} \frac{\left|f'^2 + f''^2 - 1\right|^2}{|f + f''|^2} \\ &= \frac{f'(t) + f'''(t)}{(f(t) + f''(t))^2}. \end{aligned}$$

Integration gives

$$f''(t)+f(t)=-\frac{1}{Kt+c}, \tag{2.3}$$

where c is a real constant. Then the Laplace transform gives

$$\left[s^2Y(s)-sf(0)-f'(0)\right]+Y(s)=-L\left\{G(t)\right\}=-g(s),$$

where $Y(s)$ and $G(s)$ denote the Laplace transform of $f(t)$ and $g(t)$ respectively, and $f(0)$ and $f'(0)$ are arbitrary constants. Hence

$$\begin{aligned} Y(s) &= f(0)\frac{s}{s^2+1}+f'(0)\frac{1}{s^2+1}-\frac{1}{s^2+1}L\left\{G(t)\right\} \\ &= f(0)\frac{s}{s^2+1}+f'(0)\frac{1}{s^2+1}-\frac{1}{s^2+1}g(s) \end{aligned}$$

and therefore

$$f(t)=f(0)\cos t+f'(0)\sin t-\sin(t)*G(t),$$

where the function denoted by $\sin(t)*G(t)$ and defined as

$$\sin(t)*G(t)=\int_0^t G(u)\sin(u-t)du$$

is called the convolution of the functions $\sin t$ and $G(t)$ or

$$f(t)=f(0)\cos t+f'(0)\sin t+\int_0^t G(u)\sin(u-t)du.$$

Theorem 1. Let $\alpha(t)$ be elliptic cylindrical Tzitzeica curve, then the curve $\alpha(t)$ are spacelike, timelike and null curve if and only if $f'^2(0)<1$, $f'^2(0)>1$ and $f'^2(0)=1$, respectively.

Proof. Since the curve $\alpha(t)=(\cos t,\sin t,f(t))$, the tangent of the curve is $T=\alpha'(t)=(-\sin t,\cos t,f'(t))$. The taylor series of the function f in the neighbourhood of zero is

$$f(t)=f(0)-f'(0)t+\frac{f''(0)}{2!}t^2+\frac{f'''(0)}{3!}t^3+\cdots$$

We take into consideration satisfying

$$f(0)=f'(0)\neq 0,\quad 0=f''(0)=f'''(0)=\cdots \tag{2.4}$$

in the neighbourhood of zero. Then we have

$$f(0)\cos t+f'(0)\sin t+\int_0^t \frac{\sin(u-t)}{Ku+c}du=f(0)+f'(0)t.$$

From the last equation, we get

$$\begin{aligned} f(0)\cos t &= f(0), \\ f'(0)\sin t &= f'(0)t, \\ \int_0^t \frac{\sin(u-t)}{Ku+c}du &= 0. \end{aligned}$$

Then, for $t \to 0$

$$\begin{aligned} \cos t &= 1, \\ \sin t &= t, \\ \underset{t\to 0}{Lim} \int_0^t \frac{\sin(u-t)}{Ku+c} du &= 0. \end{aligned}$$

Thus, satisfying Eq.(2.4) as $t \to 0$, $\sin t = t$ and the function f is written such as

$$f(t) = f'(0) \sin t.$$

If we take derivative of the last equation for t and square, we have

$$\begin{aligned} f'^2(t) &= f'^2(0) \cos^2 t, \\ \langle T(t), T(t) \rangle_L &= 1 - f'^2 > 0, \end{aligned}$$

or

$$\begin{aligned} (f')^2 (t) &< 1, \\ (f')^2 (0) \cos^2 t &< 1. \end{aligned}$$

Since $|\cos t| < 1$, from the last equation, we have

(i) The curve $\alpha(t)$ is spacelike curve if and only if

$$\langle T(t), T(t) \rangle_L = 1 - f'^2 > 0.$$

Then, we have

$$(f')^2 (0) < 1.$$

(ii) The Tzitzeica curve $\alpha(t)$ is timelike curve if and only if

$$\langle T(t), T(t) \rangle_L = 1 - f'^2 < 0.$$

Then, we have

$$f'^2(0) > 1.$$

(iii) The Tzitzeica curve $\alpha(t)$ is null curve if and only if

$$\langle T(t), T(t) \rangle_L = 1 - f'^2 = 0.$$

Then, we have

$$f'^2(0) = 1.$$

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On n-Fréchet spaces

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Abstract In this paper, we introduce the notion of n-sequential closure operators and use these operators to construct classes of n-Fréchet spaces as generalizations of the class of Fréchet spaces. Then we get some necessary and sufficiency conditions for a sequential space to be a Fréchet space. These results are generalizations of the preceding one of W. C. Hong in [5].

Keywords Fréchet, n-Fréchet, sequential, sequential closure, n-sequential closure.

§1. Introduction and results

M. Venkataraman poses the following problem in [8].

Problem 1.1. Characterize "the class of topological spaces which can be specified completely by knowledge of their convergent sequences".

It is well known and useful fact that every first-countable space falls into this class. To solve Problem 1.1, S. P. Franklin introduced notions of Fréchet spaces and sequential spaces ([1], [2] and [3]). Recently, W. C. Hong has investigated the relation between Fréchet spaces and sequential spaces under the sequential closure operator c_1 [4],

$$c_1(A) = \big\{x \in X : \text{ there exists a sequence } \{x_n : n \in \mathbb{N}\} \subset A, x_n \to x\big\},$$

for all subset A of X.

Recall the following notions.

Definition 1.2. ([2], [7]) Let (X, c) be a topological space endowed with the closure operator c.

(1) X is a Fréchet space, if for all $A \subset X$, $c(A) = c_1(A)$.

(2) X is a sequential space, if for all $A \subset X$, $A = c(A)$ whenever $A = c_1(A)$.

(3) X has countable tightness, if for all $A \subset X$, $x \in c(C)$ for some countable subset C of A whenever $x \in c(A)$.

From the fact that the closure operator of Fréchet spaces is specified completely by knowledge of their convergent sequences, it is natural to ask whether we can generalize the class of Fréchet spaces under this property. That is, we are interested by the following question.

Question 1.3. Is there a class of spaces which contains properly the class of Fréchet spaces such that the closure operator of spaces in this class is specified completely by knowledge of their convergent sequences?

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In this paper, we introduce the notion of n-sequential closure operators c_n, $n \in \mathbb{N}$, and use these operators to give classes of n-Fréchet spaces as generalizations of the class of Fréchet spaces. Then, we answer affirmatively Question 1.3. As the application, we give necessary and sufficiency conditions for a sequential space to be a Fréchet space. These results are generations of the preceding one of W. C. Hong in [5].

Assume that all spaces are Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers, and ω denotes $\mathbb{N} \cup \{0\}$. For terms which are not defined here, please refer to [1].

§2. Fréchet spaces

In this section we recall some results on Fréchet spaces.

Proposition 2.1. ([1], Theorem 1.6.14). Every first-countable space is a Fréchet space and every Fréchet space is a sequential space.

Proposition 2.2. ([7]). Ever sequential space has countable tightness.

Definition 2.3. ([1]). Let $f : X \longrightarrow Y$ be a map.

(1) f is a pseudo-open map, if $y \in \text{int} f(U)$ whenever $f^{-1}(y) \subset U$ with U open in X.

(2) f is a quotient map, if U open in Y whenever $f^{-1}(U)$ open in X.

Example 2.4. There exists a sequential space without being Fréchet.

Proof. Recall the Arens' space S_2 [6]. Let $T_0 = \{a_n : n \in \mathbb{N}\}$ be a sequence converging to $a \notin T_0$, and T_n, $n \in \mathbb{N}$, be a sequence converging to $b_n \notin T_n$. Put $T = \bigoplus_{n\in\mathbb{N}}(T_n \cup \{b_n\})$. Then $S_2 = \{a\} \cup (\bigcup_{n\in\omega} T_n)$ is a quotient space of $(T_0 \cup \{a\}) \oplus T$ by identifying each $b_n \in T$ to $a_n \in T_0$. We get that S_2 is a sequential space without being Fréchet.

Proposition 2.5. ([2], Proposition 2.1).

(1) Every subspace of a Fréchet space is a Fréchet space.

(2) The disjoint topological sum of any family of Fréchet spaces is a Fréchet space.

Proposition 2.6. ([2], Proposition 2.3). Let $f : X \longrightarrow Y$ be a quotient map from a Fréchet space X onto Y. Then f is pseudo-open if and only if Y is Fréchet.

Corollary 2.7. Fréchet spaces are preserved by pseudo-open maps, particularly, by closed or open maps.

Proposition 2.8. ([3], Proposition 7.2). A sequential space is Fréchet if and only if it is hereditarily sequential.

Example 2.9. ([3], Example 7.4). The product of two Fréchet spaces can be sequential without being Fréchet.

Proposition 2.10. ([5], Theorem 2). A sequential space X is Fréchet if and only if $c_1(A) = c(A)$ for every countable subset A of X.

Proposition 2.11. ([2], Proposition 2.4) Every Fréchet space is precisely the pseudo-open image of a topological sum of convergent sequences.

Corollary 2.12. The following are equivalent for a space X.

(1) X is a Fréchet space,

(2) X is a pseudo-open image of a locally compact metric space,

(3) X is a pseudo-open image of a locally separable metric space,

(4) X is a pseudo-open image of a metric space.

§3. n-Fréchet spaces

Definition 3.1. Let (X, c) be a topological endowed with the closure operator c. A map c_n, $n \in \mathbb{N}, n \geq 2$, defines by induction on n:

$$c_n(A) = c_1(c_{(n-1)}(A)),$$

for all $A \subset X$ to be called an n-sequential closure operator. The sequential closure operator c_1 is also called an 1-sequential closure operator for convince.

Proposition 3.2. Let (X, c) be a topological endowed with the closure operator c. Then the following hold for all $n \in \mathbb{N}$.

(1) $c_n(\emptyset) = \emptyset$.

(2) $A \subset c_1(A) \subset \cdots \subset c_n(A) \subset c(A)$ for all $A \subset X$.

(3) $c_n(A) \subset c_n(B)$ for all $A \subset B \subset X$.

(4) $c_n(A \cup B) = c_n(A) \cup c_n(B)$ for all $A, B \subset X$.

Definition 3.3. Let (X, c) be a topological space endowed with the closure operator c, and $n \in \mathbb{N}$. Then X is called an n-Fréchet space, if $c(A) = c_n(A)$ for all $A \subset X$.

Remark 3.4. (1) An 1-Fréchet space is precisely a Fréchet space.

(2) By Proposition 3.2, $c_n(A) \subset c(A)$ for all $A \subset X$. Then, to prove that X is an n-Fréchet, it suffices to show that $c(A) \subset c_n(A)$ for all $A \subset X$.

Next, we construct i-spaces and i-sequences by induction on $i \in \mathbb{N}$ to consider the relation between n-Fréchet spaces.

Definition 3.5. Let X be a topological space.

(1) X is called an 1-space, if $X = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ with $x_n \to x \notin \{x_n : n \in \mathbb{N}\}$, and all x_n's being distinct. Here x_n, $n \in \mathbb{N}$, is called a 0-limit point of X, and x is called an 1-limit point of X.

(2) For each $i \in \mathbb{N}, i \geq 2$, X is called an i-space, if $X = \{x\} \cup (\bigcup_{n \in \omega} X_n)$ is a quotient space obtained from a topological sum $(X_0 \cup \{x\}) \oplus \left(\bigoplus_{n \in \mathbb{N}}(X_n \cup \{x_n\})\right)$ by identifying each $y_n \in X_n$ to $x_n \in X_0$, where $X_n, n \in \mathbb{N}$, is an $(i-1)$-space with an $(i-1)$-limit point y_n, and $X_0 = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is an 1-space with an 1-limit point x. Here x is called an i-limit point of X, and a j-limit point of X_n's, $j = 0, \ldots, i-1$, is called a j-limit point of X.

Remark 3.6. (1) Each 1-space is an infinite convergent sequence, and each 2-space is an Aren's space S_2 (see Example 2.4).

(2) For each $i \in \mathbb{N}$, an i-space is an infinite countable set.

Definition 3.7. Let X be a topological space and $S \subset X$.

(1) S is called an 1-sequence of X, if $S = \{x_n : n \in \mathbb{N}\}$ is a sequence converging to x. We say that S converges under 1 to x, and write $S \xrightarrow{1} x$, here x is called an 1-limit point of S.

(2) For $i \in \mathbb{N}, i \geq 2$, S is called an i-sequence of X, if $S = \bigcup_{n \in \mathbb{N}} L_n$, where L_n is an $(i-1)$-sequence, $L_n \xrightarrow{i-1} x_n$ for each $n \in \mathbb{N}$, and $L_0 = \{x_n : n \in \mathbb{N}\}$ is an 1-sequence with $L_0 \xrightarrow{1} x$. We say that S converges under i to x, and write $S \xrightarrow{i} x$, here x is called an i-limit point of S.

Remark 3.8. The following hold.

(1) The set S of all 0-limit points of an i-space X is an i-sequence satisfying that $c_i(S) = X$.

(2) Each i-sequence is a countable set.

(3) Every finite set is an i-sequence. But the following Example 3.9 proves that there is an infinite countable subset of a metric space without being an i-sequence.

Example 3.9. There is an infinite countable subset of a metric space without being an i-sequence.

Proof. Let $\mathbb{R}$ be a discrete space. Then $\mathbb{R}$ is a metric space. We have that $\mathbb{N}$ is an infinite countable subset of $\mathbb{R}$, and $\mathbb{N}$ is not an i-sequence. In fact, if there exists an $i \in \mathbb{N}$ such that $\mathbb{N}$ is an i-sequence. Since each 1-sequence of $\mathbb{R}$ with the discrete topology is finite, $\mathbb{N}$ is finite. It is a contradiction.

Lemma 3.10. Let X be a topological space, and $A \subset X$. Then $x \in c_n(A)$ if and only if there exists an n-sequence $S \subset A$ such that $S \xrightarrow{n} x$.

Proof. Necessity. For $n = 1$, we get that $x \in c_1(A)$. Then there exists a sequence $S = \{x_n : n \in \mathbb{N}\} \subset A, x_n \to x$. It is clear that S is an 1-sequence, $S \subset A$, and $S \xrightarrow{1} x$.

For each $n \geq 2$, we get that $x \in c_n(A) = c_1(c_{(n-1)}(A))$. Then there exists an 1-sequence $L \subset c_{(n-1)}(A)$ such that $L \xrightarrow{1} x$. Put $L = \{x_i : i \in \mathbb{N}\}$ then $x_i \in c_{(n-1)}(A)$ for all $i \in \mathbb{N}$. Note that for each $i \in \mathbb{N}$, there is an $(n-1)$-sequence $L_i \subset A$ such that $L_i \xrightarrow{n-1} x_i$. Then $S = \bigcup_{i \in \mathbb{N}} L_i \subset A$ is an n-sequence, and $S \xrightarrow{n} x$.

Sufficiency. Since $S \xrightarrow{n} x$, $x \in c_n(S)$. By Proposition 3.2, $x \in c_n(A)$

Corollary 3.11. Let X be an n-Fréchet space, and $A \subset X$. Then $x \in c(A)$ if and only if there exists an n-sequence $S \subset A$ such that $S \xrightarrow{n} x$.

Proposition 3.12. The following hold for a space X.

(1) If X is an n-Fréchet space, then X is an $(n+1)$-Fréchet space.

(2) If X be an n-Fréchet space, then X is a sequential space.

Proof. (1) Let $A \subset X$ and $x \in c(A)$. Since X is n-Fréchet, $x \in c_n(A)$. Note that $c_n(A) \subset c_{(n+1)}(A)$. Then $x \in c_{(n+1)}(A)$. So $c(A) \subset c_{(n+1)}(A)$, i.e., X is an $(n+1)$-Fréchet space.

(2) Let $A \subset X$ and $A = c_1(A)$. We need to prove that $A = c(A)$. It follows from $A = c_1(A)$ that $A = c_1(A) = ... = c_{(n-1)}(A) = c_n(A)$. Since $c(A) = c_n(A)$, $A = c(A)$.

Corollary 3.13. Every n-Fréchet space has countable tightness.

The following examples to prove that inverse implications in Proposition 3.12 do not hold.

Example 3.14. For all $n \in \mathbb{N}$, there exists an $(n+1)$-Fréchet space which is not an n-Fréchet space.

Proof. For each $n \in \mathbb{N}$, let X be an $(n+1)$-space.

(i) X is an $(n+1)$-Fréchet space.

Let $A \subset X$ and $x \in c(A)$. We shall prove that $x \in c_{(n+1)}(A)$.

If x is a 0-limit point, then x is an isolated point. Then from $x \in c(A)$ we get $x \in A \subset c_{(n+1)}(A)$.

If x is an 1-limit point, then x has a countable neighborhood system. Then from $x \in c(A)$ we get that there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset A$ such that $x_n \to x$. Hence $x \in c_1(A) \subset c_{(n+1)}(A)$.

If x is an i-limit point, $i = 2, \ldots, n+1$, then there exist $(i-1)$-spaces Y_m's, $m \in \mathbb{N}$, with $(i-1)$-limit points y_m's, and an 1-sequence $Y_0 = \{y_m : m \in \mathbb{N}\}$ with the 1-limit point x such that $\{x\} \cup (\bigcup_{n\in\omega} Y_n) \subset X$, and $\{x\} \cup (\bigcup_{m\in\omega} Y_m)$ is an i-space. Sice $x \in c(A)$, there is infinitely numbers $m \in \mathbb{N}$ such that $A \cap Y_m$ is an $(i-1)$-sequence with the $(i-1)$-limit point y_m. Then there is infinitely numbers m such that $y_m \in c_{(i-1)}(A \cap Y_m)$. Hence $x \in c_i(A \cap Y_m) \subset c_i(A)$. It implies that $x \in c_i(A) \subset c_{(n+1)}(A)$.

(ii) X is not an n-Fréchet space.

Let

$$A = \{x \in X : x \text{ is the 0-limit point of } X\}.$$

Then $c_n(A) = X - \{a\}$ with a is the $(n+1)$-limit point of X. On the other hand, $a \in c(A)$. Therefore $c_n(A) \neq c(A)$. So X is not n-Fréchet.

Example 3.15. There is a sequential space which is not an n-Fréchet space for all $n \in \mathbb{N}$.

Proof. Let $S_\infty = \oplus_{m\in\mathbb{N}} S_m$, where S_m's are m-spaces.

(i) S_∞ is a sequential space.

It follows from the proof of Example 3.14 that each m-space S_m is an m-Fréchet space. Then each S_m is a sequential space by Proposition 3.12. So S_∞ is a sequential space by ([2], Proposition 1.6).

(ii) S_∞ is not an n-Fréchet space.

For each $n \in \mathbb{N}$, we put

$$A = \big\{x \in S_{(n+1)} : x \text{ is the 0-limit point of } S_{(n+1)}\big\}.$$

Then $c_n(A) = S_{(n+1)} - \{x\}$, here x is the $(n+1)$-limit point of $S_{(n+1)}$. On the other hand, $x \in c(A)$. Then $c(A) \neq c_n(A)$. It implies that S_∞ is not an n-Fréchet space.

Remark 3.16. It follows from Proposition 3.12, Example 3.14, and Example 3.15 that

$$\text{Fréchet} \Rightarrow n\text{-Fréchet} \Rightarrow (n+1)\text{-Fréchet} \Rightarrow \text{sequential},$$

for all $n \in \mathbb{N}$. Moreover, all of inverse implications do not hold if $n \geq 2$.

Lemma 3.17. Let $f : X \longrightarrow Y$ be a map. If S is an n-sequence in X with the n-limit point x, then $f(S)$ is an n-sequence in Y with the n-limit point $f(x)$.

Proposition 3.18. Let $f : X \longrightarrow Y$ be a map and X be an n-Fréchet space, then Y is an n-Fréchet space.

Proof. For all $A \subset Y$ and $y \in c(A)$, we shall prove that $y \in c_n(A)$. Suppose that $f^{-1}(y) \cap c\big(f^{-1}(A)\big) = \emptyset$. Then $U = X - c\big(f^{-1}(A)\big)$ is an open set containing $f^{-1}(y)$. Since f is a pseudo-open map and $f^{-1}(y) \subset U$, $y \in \mathrm{int}\big(f(U)\big) = \mathrm{int}\Big(f(X - c\big(f^{-1}(A)\big)\Big) = \mathrm{int}\Big(Y - f\big(c\big(f^{-1}(A)\big)\big)\Big) \subset \mathrm{int}(Y - A) = Y - c(A)$. So $y \notin c(A)$. It is a contradiction. Then there exists some point $x \in f^{-1}(y) \cap c\big(f^{-1}(A)\big)$. Since X is an n-Fréchet space, there is an n-sequence $S \subset f^{-1}(A)$ such that $S \xrightarrow{n} x$ by Corollary 3.11. On the other hand, $y = f(x)$ is the n-limit point of n-sequence $f(S) \subset f\big(f^{-1}(A)\big) = A$ by Lemma 3.17. It implies that $y \in c_n(A)$.

Example 3.19. For all $n \geq 2$, there exists an n-Fréchet space without being hereditarily n-Fréchet.

Proof. For $n \geq 2$, let X be an n-space. Then X is an n-Fréchet space by the proof of Example 3.14. We shall prove that X is not hereditarily n-Fréchet. Conversely, suppose that X is hereditarily n-Fréchet. By Proposition 3.12, X is a hereditarily sequential space. It follows from Proposition 2.8 that X is an 1-Fréchet space. It is a contradiction (see Example 3.14). Then X is not hereditarily n-Fréchet.

Proposition 3.20. Every closed subspace of an n-Fréchet space is an n-Fréchet space.

Proof. Let X be an n-Fréchet space, $Y \subset X, Y = c(Y)$ and $A \subset Y$. Let c^Y and c_n^Y be the closure operator and the n-sequential closure operator of the subspace Y, respectively. We need to prove that $c^Y(A) = c_n^Y(A)$. Since $c_n(A) = c(A) \subset c(Y) = Y$, $c_n(A) = c_n^Y(A)$. On the other hand, $c^Y(A) = c(A) \cap Y = c(A) \cap c(Y) = c(A) = c_n(A)$. Then $c^Y(A) = c_n^Y(A)$.

Proposition 3.21. The disjoint topological sum of any family of n-Fréchet spaces is an n-Fréchet space.

Proof. Let $\{X_i : i \in I\}$ be a disjoint collection of n-Fréchet spaces, and $X = \bigoplus_{i\in I} X_i$. For any $A \subset X$ and $x \in c(A)$, we need to prove that $x \in c_n(A)$. Since $x \in c(A)$, there exists some $i \in I$ such that $x \in c(A) \cap X_i$. Then $x \in c^{X_i}(A) = c_n^{X_i}(A)$, here c^{X_i} and $c_n^{X_i}$ are the closure operator and n-sequential closure operator of X_i, respectively. Since $c_n^{X_i}(A) \subset c_n(A)$, $x \in c_n(A)$.

Proposition 3.22. A space having the countable tightness X is an n-Fréchet space if and only if $c(A) = c_n(A)$ for every countable subset $A \subset X$.

Proof. Necessary. It is clear.

Sufficiency. Let $c_n(A) = c(A)$ for every countable subset $A \subset X$. For any $B \subset X$ we need to prove that $c(B) = c_n(B)$. For any $x \in c(B)$, since X has the countable tightness, there is a countable subset $C \subset B$ such that $x \in c(C)$. Note that $c(C) = c_n(C)$ by countability of C. Then $x \in c_n(C) \subset c_n(B)$. It implies that $c(B) = c_n(B)$.

Corollary 3.23. The following hold for a space X.

(1) Under X being a sequential space, then X is an n-Fréchet space if and only if $c(A) = c_n(A)$ for every countable subset $A \subset X$.

(2) Under X being an m-Fréchet space and $m > n$, then X is an n-Fréchet space if and only if $c_m(A) = c_n(A)$ for every countable subset $A \subset X$.

Proof. It is straightforward from Proposition 2.2, Corollary 3.13, and Proposition 3.22.

Remark 3.24. For $n = 1$ in Corollary 3.23.(1) we get ([5], Theorem 2). Then, Proposition 3.22 is a generation of ([5], Theorem 2) (see Proposition 2.10).

Proposition 3.25. A sequential space X is an n-Fréchet space if and only if $c_n(S) = c_{(n+1)}(S)$ for every $(n+1)$-sequence $S \subset X$.

Proof. Necessary. It is straightforward from Proposition 3.21.

Sufficiency. Conversely, suppose that X is not n-Fréchet. Then there exists $A \subset X$ such that $c_n(A) \neq c(A)$. Since $A \subset c_n(A) \subset c(A)$, $c(A) \subset c\big(c_n(A)\big) \subset c\big(c(A)\big)$. Then $c\big(c_n(A)\big) = c(A)$. It implies that $c_n(A)$ is not closed in a sequential space X. So $c_n(A) \neq c_1\big(c_n(A)\big)$. Hence there is some $x \in c_1\big(c_n(A)\big)$ such that $x \notin c_n(A)$. Because $x \in c_1\big(c_n(A)\big) = c_{(n+1)}(A)$, there is an $(n+1)$-sequence $S \subset A$ such that $S \overset{n+1}{\to} x$ by Lemma 3.10. Then $x \in c_{(n+1)}(S) = c_n(S) \subset c_n(A)$. It implies that $x \in c_n(A)$. It is a contradiction.

Corollary 3.26. A sequential space X is an 1-Fréchet space if and only if $c_1(S) = c_2(S)$

for every 2-sequence $S \subset X$.

Corollary 3.27. A space X is an 1-Fréchet space if and only if $c_1(S) = c(S)$ for every 2-sequence $S \subset X$.

Proof. Necessary. It is obvious.

Sufficiency. It follows from Proposition 3.2 that $c_1(S) = c_2(S)$. Then X is an 1-Fréchet space by Corollary 3.26.

Remark 3.28. It follows from Example 3.9 that the collection of n-sequences is a proper subset of the collection of countable subsets of a sequential space (indeed, a metric space). So, Corollary 3.27 and Corollary 3.26, are generations of ([5], Theorem 2) (see Proposition 2.10).

It is well known that the product of a collection of Fréchet spaces is not Fréchet (see Example 2.9). So the following question rises naturally.

Question 3.29. Is the product of a collection of n-Fréchet spaces an n-Fréchet space?

Note that the answer of Question 3.29 is negative with Hausdorff assumption (see [1,2.3 K]).

On the other hand, every Fréchet space and sequential space is precisely the pseudo-open image and the quotient image of a metric space, respectively (see Corollary 2.12 and ([2], Corollary 1.14), and the class of n-Fréchet spaces is between the class of Fréchet spaces the class of sequential spaces (see Remark 3.16). So we have the following question.

Question 3.30. What maps is every n-Fréchet space precisely image of a metric space under?

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Product of powers of quasinormal composition operators

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Abstract Let T_i, $i = 1, 2$ be measurable transformations which define bounded composition operators C_{T_i} on L^2 of a σ-finite measure space. Denote their respective Radon-Nikodym derivatives by h_i, $i = 1, 2$. The main result of the paper is that, if $h_i \circ T_i = h_j$ a.e., for $i, j = 1, 2$, then for each of the positive integers m, n, p the operators $(C^m_{T_1} C^n_{T_2})^p$ and $(C^m_{T_2} C^n_{T_1})^p$ are quasinormal. As a consequence, we see that the sufficient condition established in our paper [5] for quasinormality of a composition operator is actually sufficient for all powers to be quasinormal.

Keywords Quasinormal operators, powers of quasinormal composition operators.

§1. Preliminaries

Let (X, Σ, λ) be a sigma-finite measure space and let T be a measurable transformation from X into itself. Set $L^2 = L^2(X, \Sigma, \lambda)$. The equation $C_T f = f \circ T$ for every $f \in L^2$ defines a composition transformation from L^2 to the space of C-valued functions on X. C_T is a bounded linear operator on L^2 precisely when (i) $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ and (ii) $h = d(\lambda \circ T^{-1})/d\lambda$ is in $L^\infty(X, \Sigma, \lambda) = L^\infty$. Denote by $R(C_T)$ the range of C_T, by C^*_T the adjoint of C_T, and define $T^{-1}(\Sigma)$ as the relative completion of the σ - algebra $\{T^{-1}(\text{A})$: $A \in \Sigma\}$.

The following lemmas, due to Harrington and Whitley[2], is well known and useful.

Lemma 1.1. Let P denotes the projection of L^2 onto $\overline{R(C_T)}$.

(i) $C^*_T C_T f = hf$ and $C_T C^*_T f = (h \circ T)Pf$, for all $f \in L^2$. (1)

(ii) $\overline{R(C_T)} = \{f \in L^2 : f \text{ is } T^{-1}(\Sigma) \text{ measurable }\}$. (2)

(iii) If f is $T^{-1}(\Sigma)$ measurable and g and fg belongs to L^2, then $P(fg) = fP(g)$. (3)

Lemma 1.2. With h, T and P as above,

$$< (h^n \circ T)Pg, g >=< (h^n \circ T)g, g >, n \in N, g \in L^2.$$

Let $B(H)$ denote the Banach algebra of all bounded linear operators on a Hilbert space H. An operator A on a complex Hilbert space H is said to be quasinormal if A commutes with A^*A.

Theorem 1.3. Let C_T, $M_\theta \in B(L^2(\lambda))$. Then $C_T M_\theta = M_\theta C_T$ if and only if $\theta = \theta \circ T$ a.e., where M_θ is the multiplication operator induced by θ.

Theorem 1.4. Let C_{T_1}, $C_{T_2} \in B(L^2(\lambda))$ be quasinormal with $\mathrm{h}_1 \circ \mathrm{T}_2 = \mathrm{h}_2$ a.e. Then the products $\mathrm{C}_{T_1}\mathrm{C}_{T_2}$ and C_{T_2}, C_{T_1} are quasinormal.

§2. Main result and corollaries

In order to prove our main result, it is necessary to state and prove several lemmas. They are stated so that each lemma depends on some subset of the previously stated ones. The proofs which are given conatin the essential ideas and may easily be adjusted to give the ones we omit.

Lemma 2.1. If $h \circ T = h$ a.e., then for $n \in N$, $f \in L^2$, we have
$$< h^n f, f >=< (C_T^n)^* C_T^n f, f >. \tag{4}$$

Proof. For n = 1, the lemma is true by (1). Suppose (4) holds for $n = 1, 2, \cdots k$ and all $f \in L^2$. Then $< h^k f, f >=< (C_T^k)^* C_T^k f, f >$.

By induction hypothesis, $< h^{k+1} f, f >=< C_T^* C_T (C_T^* C_T)^k f, f >$
$$= < h h^k f, f >=< (C_T^* C_T)^{k+1} f, f >,$$
so that (4) holds for n = k + 1 and lemma 2.1 follows by induction.

Lemma 2.2. If $h \circ T = h$ a.e. , then for all $n \in N$, $f \in L^2$, we have
$< C_T^n \ (C_T^n)^* f, f >=< (h \circ T)^n f, f >=< h^n f, f >$.

Proof. For $n = 1, < C_T C_T^* f, f >=< (h \circ T) f, f >=< hf, f >$.

Since $< C_T^* C_T f, f >=< (h \circ T) P f, f >= \int (h \circ T) |f|^2 dm$
$$= \int \mathrm{h}\ |\mathrm{f}|^2\ \mathrm{dm} = < \mathrm{hf}, \mathrm{f} >.$$

Since h∘T = h a.e. By induction hypothesis we can prove lemma 2.2.

Now set $A = C_{T_1}$, $B = C_{T_2}$, so that the product AB is the operator C_{T_3}.

Lemma 2.3. With A, B, h_1 and h_2 as above, if (a) $h_2 \circ T_2 = h_1$ a.e. (b) $h_1 \circ T_1 = h_1$ a.e., then for each $m, n \in N, f \in L^2$, we have $< (A^m B^n)^* (A^m B^n) f, f >=< h_2^{m+n} f, f >$.

Proof. First we prove

Claim 2.3.1. With T and h as in §1, for all $r, m \in N$, $f \in L^2$,
we have $< (h \circ T)^r C_T^m f, C_T^m f >=< h^{r+m} f, f >$. (5)

Proof of claim 2.3.1.

Fix r and induct on m. For $m = 1$, $f \in L^2$
$$< (h \circ T)^r C_T f, C_T f >= \int (h \circ T)^r (|f|^2 \circ T) dm$$
$$= \int (h^r |f|^2) h dm =< h^{r+1} f, f >.$$

Suppose (5) holds for m = k and for all $f \in L^2$.

Then $< (h \circ T)^r C_T^{k+1} f, C_T^{k+1} f >=< (h \circ T)^{r+k} C_T f, C_T f >$
$$= \int (h^{r+k} \circ T)(|f|^2 \circ T) dm =< h^{r+k+1} f, f >.$$

So (5) holds for $m = k + 1$. The claim 2.3.1. is proved by induction.

To finish the proof of lemma 2.3, observe that
$$< (A^m B^n)^* (A^m B^n) f, f >=< (A^m)^* A^m B^n f, B^n f >$$
$$= < h_1^m B^n f, B^n f) \text{ (by Lemma 2.1)}$$

$= < (h_2 \circ T_2)^m B^n f, B^n f) >$ (by hypothesis)

$= < h_2^{m+n} f, f >$ (by claim 2.3.1)

Lemma 2.4. With A, B, h_1 and h_2 as above, if $h_i \circ T_i = h_j, i, j = 1, 2,$ (6)

then for each $m, n \in N, f \in L^2$, we have $< (A^m B^n)(A^m B^n)^* f, f >=< h_2^{m+n} f, f >$.

Proof. First we prove

Claim 2.4.1. If h $\circ T = h$ a.e., then for all $r, m \in N, f \in L^2$, we have

$< h^r (C_T^m)^* f, (C_T^m)^* f >=< (h \circ T)^{r+m} f, f >$.

Proof of claim. Fix r and induct on m. For $m = 1$ and $f \in L^2$,

$< h^r C_T^* f, C_T^* f >=< (h \circ T)^r C_T C_T^* f, f >$

$= < (h^{r+1} \circ T) P f, f >$ (by lemma 1.1)

$= < ((h^{r+1} \circ T) f, f >$ (by lemma 1.2)

By induction hypothesis claim 2.4.1 can be proved.

Now $< (\mathrm{A}^m \mathrm{B}^n)\ (\mathrm{A}^m \mathrm{B}^n)^*$ f, f $>$ $=$ $< [B^n (B^n)^*](A^m)^* f, (A^m)^* f >$

$= < (h_2 \circ T_2)^n (A^m)^* f, (A^m)^* f >$ (by lemma 2.2)

$= < h_1^n (A^m)^* f, (A^m)^* f >$ (by hypothesis)

$= < (h_1 \circ T_1)^{m+n} f, f >$ (by claim 2.4.1)

$= < h_2^{m+n} f, f >$ (by hypothesis)

Finally, similar technique and the above lemmas may be used to prove the following pair of results, which we collect as Lemma 2.5 and state without proof.

Lemma 2.5. If (6) holds, then for all $m, n, p \in N, f \in L^2$, we have

$$< [(A^m B^n)^p]^* [(A^m B^n)^p] f, f >=< h_2^{(m+n)p} f, f > \tag{7}$$

and

$$< (A^m B^n)^p [(A^m B^n)^p]^* f, f >=< h_2^{(m+n)p} f, f > . \tag{8}$$

Remark. $\mathrm{h}_2 \circ \mathrm{T}_2 = \mathrm{h}_2$ is not necessary in the proof of (7). Now we may easily prove our main result.

Lemma 2.6. Let $C_T \in B(L^2(\lambda))$ be quasinormal. Then $M_h^n C_T = C_T M_h^n, n \in$N.

Lemma 2.7. Let C_{T_i}, C_{T_j} are quasinormal with $h_i \circ T_j = h_i$ a.e., $h_j \circ T_i = h_j$ a.e. Then $M_{h_i}^m C_{T_j}^n = C_{T_j}^n M_{h_i}^m$.

In this section, we shall show that the sufficient condition established in the Theorem 2.2[5] for quasinormality of a composition operator is actually sufficient for all powers to be quasinormal.

Theorem 2.8. Let C_{T_i} be quasinormal with $h_i \circ T_j = h_i$ a.e. for $i, j = 1, 2$. Then $(C_{T_1}^m C_{T_2}^n)^p$ and $(C_{T_2}^m C_{T_1}^n)^p$ are also quasinormal for all $m, n, p \in N$.

Proof.

Now

$(C_{T_1}^m C_{T_2}^n)((C_{T_1}^m C_{T_2}^n))^* (C_{T_1}^m C_{T_2}^n) = C_{T_1}^m C_{T_2}^n C_{T_2}^{n*} ((C_{T_1}^{m*} C_{T_1}^m) C_{T_2}^n$

$$= C_{T_1}^m C_{T_2}^n C_{T_2}^{n*} M_{h_1}^m C_{T_2}^n$$

$$= C_{T_1}^m C_{T_2}^n (C_{T_2}^{n*} C_{T_2}^n) M_{h_1}^m$$

$$= C_{T_1}^m C_{T_2}^n M_{h_2}^n M_{h_1}^m$$

$$= C_{T_1}^m M_{h_2}^n C_{T_2}^n M_{h_1}^m$$

$$= M_{h_2}^n C_{T_1}^m C_{T_2}^n M_{h_1}^m$$

$$= C_{T_2}^{n*} C_{T_2}^n C_{T_1}^m C_{T_2}^n M_{h_1}^m$$

$$= C_{T_2}^{n*} C_{T_2}^n C_{T_1}^m M_{h_1}^m C_{T_2}^n$$

$$= C_{T_2}^{n*} C_{T_2}^n M_{h_1}^m C_{T_1}^m C_{T_2}^n$$

$$= C_{T_2}^{n*} M_{h_1}^m C_{T_2}^n C_{T_1}^m C_{T_2}^n$$

$$= C_{T_2}^{n*} C_{T_1}^{m*} C_{T_1}^m C_{T_2}^n C_{T_1}^m C_{T_2}^n$$

$$= (C_{T_1}^m C_{T_2}^n))^* (C_{T_1}^m C_{T_2}^n)(C_{T_1}^m C_{T_2}^n).$$

This implies $C_{T_1}^m C_{T_2}^n$ is a quasinormal composition operator. And similarly $C_{T_2}^n C_{T_1}^m$ is a quasinormal composition operator.
Therefore $(C_{T_1}^m C_{T_2}^n)^p$ and $(C_{T_2}^m C_{T_1}^n)^p$ are also quasinormal composition operators.

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Smarandache magic square

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Abstract This paper contains a magic square. A square array of natural numbers in which the sum of each row and each column is same is a magic square. Smarandache magic square has been defined by Sabin Tabirca [1].

Keywords Magic square.

A Smarandache magic square(SMS) is a square array containing $S(i)$, the Smarandache numbers, only in n rows and m columns such that the sum of each row and each column is same.

The difference between ordinary magic square and SMS is that the elements of SMS are of the form: $S(1)$, $S(2)$, $S(3)$, $\ldots$, $S(n^2)$.

Let (a_{ij}) form the SMS, defined as:

1. $[(a_{ij}), i = 1$ to $n, j = 1$ to $n] = [\mathrm{S(i)}, i = 1$ to $n^2]$;

2. $\sum a_{ij} = C$, $j = 1$ to n;

3. $\sum a_{ij} = C, i = 1$ to n;

4. $\sum S(i) = n \cdot C$, where C is the value of the determinant formed by this SMS.

S. Tabirca has claimed that the SMS exists only for the numbers $6, 7, 9, 58$ and 59. The other numbers from $n = 2$ to 100 do not form the SMS. The reason is that the fourth criterion above, i.e. $\sum S(i) = n \cdot C$ is not satisfied.

The following is the table for n and $\sum S(i)$ for which the SMS exists.

n	$\sum S(i)$
6	330
7	602
9	1413
58	1310162
69	2506080

SMS does not exist for $n = 2, 4, 5\ldots$ because for $n = 4, \sum S(i)$, for $i = 1$ to 16 is 85, and 4 does not divide 85. Similarly for other values of n.

Here is an example of Smarandache magic square. It is of order 6.

							Sum of row elements
	3	4	11	11	23	3	55
	5	29	5	4	5	7	55
	4	13	11	9	5	13	55
	5	2	17	6	6	19	55
	7	7	7	17	10	7	55
	31	0	4	8	6	6	55
Sum of column elements	55	55	55	55	55	55	330

Now following is the magic square in the form of Smarandache functions.

$$\begin{matrix} S(3) & S(4) & S(11) & S(22) & S(23) & S(6) \\ S(5) & S(29) & S(10) & S(8) & S(15) & S(7) \\ S(12) & S(13) & S(33) & S(27) & S(20) & S(26) \\ S(30) & S(2) & S(17) & S(18) & S(36) & S(19) \\ S(14) & S(21) & S(35) & S(34) & S(25) & S(28) \\ S(31) & S(1) & S(24) & S(23) & S(16) & S(9) \end{matrix}$$

Here, $\sum\limits_{i=1}^{36} S(i) = 330$, $n = 6$ and $K =$Value of each row/ column of magic square $= 55$, and $330 = 6 \times 55$. Hence the condition $\sum\limits_{i=1}^{36} S(i) = n.K$ is satisfied.

Therefore the above square is Smarandache magic square.

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Pairwise semi compact and pairwise semi lindeloff spaces

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Abstract In this paper (i, j) semi compact and pairwise semi compact; (i, j) semi Lindeloff and pairwise semi Lindeloff Bitopological spaces are defined and their basic properties are studied.

Keywords Bitopological space, semi open set, (i, j) semi compact, pairwise semi compact, (i, j) semi Lindeloff, pairwise semi Lindeloff Bitopological space.

§1. Introduction

J.C. Kelly[2] introduced the concept of Bitopological spaces in 1963 which paved way to the theory of Bitopological spaces. After him many Authors defined different version of Bitopological spaces and many of their properties like, compactness, connectedness, countablity and separation properties were studied with respect to different type of open sets namely semi open, pre open and semi pre (β) open sets. Norman Levine[5] introduced the concept of semi open sets and semi continuity in topological spaces, Maheswari[4] and prasad[4] extended the notions of semi open sets and semi continuity to Bitopological spaces. Shantha Bose[6] further investigated the properties of semi open sets and semi continuity in Bitopological spaces. Ian E. Cooke and Ivan L. Reilly[8] defined and studied the basic properties of compactness in Bitopological spaces. F. H. Khedr et.al.[3] studied interrelations between different open sets between Bitopological spaces. Recently S. Balasubramanian and G. Koteeswara Rao introduced weak and strong Lindeloff Bitopological spaces. In this paper the author introduced compactness using semi open sets in Bitopological spaces which is independent of compactness defined by others and tried to extend the concepts of Lindeloff condition and discussed basic properties in (i,j) and pairwise semi lindeloff spaces.

§2. Preliminaries

A non empty set X together with two topologies τ_1 and τ_2 is called a Bitopological space[2]. Hereafter a space X is called as a Bitopological space unless otherwise stated in this paper. A subset A of X is called (τ_i, τ_j) semi open (briefly (i, j) semi open)[3] if there exists $U \in \tau_i$ such that $U \subset A \subset Cl_j(U)$. A subset A is said to be pairwise semi open if it is (i, j) semi open and (j, i) semi open. A space X is called a pairwise compact[8] if every pairwise open cover

$U \in \tau_i \cup \tau_j$ has a finite sub cover. A space is called a weak (strong) [locally] compact if it is either τ_i or $\tau_j(\tau_i$ and $\tau_j)$ [locally] compact.

Definition 2.1.[6] A subset A of a topological space (X, τ) is said to be a semi open set if there is an open set U of X such that $U \subset A \subset cl(U)$. A function $f\colon (X,\tau) \to (Y,\sigma)$[6] is said to be semi continuous if for each $V \in \sigma$, $f^{-1}(V)$ is semi open in X.

Definition 2.2.[3] $A \subseteq (X, \tau_i, \tau_j)$ is said to be (i,j) semi open if there exists $U \in \tau_i$ such that $U \subset A \subset cl_j(U)$.

Example 1. Let X be the real plane τ_i the half open rectangle topology and τ_j be the usual topology on X. $A = \{(x, y)/0 \leq x < 1, 0 \leq y < 1\} \cup (1, 1)$ is τ_i semi open w.r.to τ_j but neither τ_i open nor τ_j open.

Example 2. Let X be the real line τ_i be the lower limit topology on X and τ_j the usual topology on X. $A = \{x/0 \leq x < 1\}$ is τ_i semi open w.r.to τ_j but neither τ_i open nor τ_j open.

Definition 2.3. Let $f\colon (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be a function then f is said to be pairwise continuous[resp. pairwise open map] if the induced functions $f : (X, \tau_i) \to (Y, \sigma_i)$ and $f : (X, \tau_j) \to (Y, \sigma_j)$ are both continuous [resp. open].

Definition 2.4.[3] Let $f\colon (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be a single valued function then f is said to be pairwise semi continuous if the inverse image of each σ_i-open set of Y is (i, j) semi open in X, where $i \neq j$, $i, j = 1, 2$.

Theorem 2.5.[3] Let A be a subset of (X, τ_i, τ_j). If there exists an (i, j) semi open set U such that $U \subset A \subset cl_j(U)$, then A is (i, j) semi open.

Theorem 2.6.[3] The union of arbitrarily many (i, j) semi open sets is (i, j) semi open.

Proof. Let $\{A_k/k \in K\}$ be a collection of (i, j) semi open sets in (X, τ_i, τ_j). For each $k \in K$, $\exists$ $a\tau_i$-open set U_k such that $U_k \subset A_k \subset cl_j(U_k)$, then $\cup U_k \subset \cup A_k \subset \cup cl_j(U_k) = cl_j(\cup U_k)$. Put $U = \cup U_k$, U is τ_i-open. Thus $\cup A_k$ is (i, j) semi open.

Theorem 2.7.[3] Let $A \subset Y \subset X$ where X is a Bitopological space. If A is (i, j) semi open in X, it is (i, j) semi open in Y.

Theorem 2.8.[3] Let $f : (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be pairwise continuous pairwise open map. If A is (i, j) semi open in X then $f(A)$ is (i, j) semi open in Y.

Proof. Let A be(i, j) semi open in X, then there exists $U \in \tau_i$ such that $U \subset A \subset cl_j(U)$. Now $f(U) \subset f(A) \subset f(cl_j(U)), f(U)$ is σ_i-open, f being open map and $f(cl_j(U)) \subset cl_j(f(U))$, f being continuous. $f(A)$ is (i, j) semi open in X.

Theorem 2.9.[3] Let $f : (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be pairwise semi continuous pairwise open map. If A is (i, j) semi open in Y then $f^{-1}(A)$ is (i, j) semi open in X.

Proof. Let A be (i, j) semi open in Y, then there exists $W \in \sigma_i$ such that $W \subset V \subset cl_j(W)$. Since f is pairwise open, it follows that $f^{-1}(W) \subset f^{-1}(V) \subset f^{-1}(cl_j(W)) \subset cl_j(f^{-1}(W))$. Since f is pairwise semi continuous, $f^{-1}(W)$ is (i, j) semi open in X. By theorem 2.5, $f^{-1}(V)$ is (i, j) semi open in X.

Remark 1. [3] τ_i open $\Rightarrow (i, j)$ semi open.

§3. (i, j) Semi compact Bitopological space

Definition 3.1. (X, τ_i, τ_j) is said to be (i) $(i,\ j)$ semi compact if each $(i,\ j)$ semi open cover has a finite (i,j) semi open sub cover. (ii) locally (i, j) semi compact if each $x \in X$ has (i, j) semi-open neighborhood with compact (i, j) semi closure.

Example 3. since every open set is (i, j) semi open, example $X = R, \tau_i = R_\iota, \tau_j = R_\Omega$; where R_Ω = minimum well ordered uncountable set will act as the example for these two definitions.

Example 4. Let $X = [a, b]$ be the subspace of real numbers with $\tau_i = \{[a, c)/a < c < b\}$ and $\tau_j = \{(c, d)/c < d\}$ as basic open sets then, (X, τ_i, τ_j) is (i, j) semi compact Bitopological space.

Theorem 3.2. Every (i, j) semi compact space is locally $(i,\ j)$ semi compact.

Theorem 3.3. Let (X, τ_i, τ_j) be Bitopological space and $A \subset (X, \tau_i, \tau_j)$. Then A is (i, j) semi compact subset of (X, τ_i, τ_j) if the subspace $(A, \tau_{i/A}, \tau_{j/A})$ is (i, j) semi compact.

Theorem 3.4. A τ_i-closed subspace of a [locally] (i, j)semi compact space is [locally] (i, j) semi compact. The product of two [locally] (i, j) semi compact spaces is [locally] (i, j) semi compact.

Proof. $A \subseteq (X, \tau_i, \tau_j)$ is τ_i-closed and (X, τ_i, τ_j) is (i, j) semi compact. For $U = \{U_i/i \in I\}$ is (i, j) semi open cover of A we can find $U^{'} = \{G(U)/U \in U\} \cup \{X - A\}$ an (i, j) semi open cover of X, where G is such that $U = G \cap A$, which implies $U^{'}$ admits finite sub cover giving rise to the finite sub covers for U. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is (i, j) semi compact.

$A \subseteq (X, \tau_i, \tau_j)$ is τ_i-closed, (X, τ_i, τ_j) is locally (i, j) semi compact and $x \in A \subseteq X$ be any point. Then$x$ has (i, j) semi compact neighborhood G. Since G is (i, j) semi compact, by theorem 3.3 we can find (i, j) semi compact neighborhood $U = G \cap A$. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is locally (i, j) semi compact.

Direct consequence of Tychnoff's theorem.

Theorem 3.5. Let (X, τ_i, τ_j) be any [locally] (i, j) semi compact Bitopological space. Then (X, τ_i) is [locally] compact.

Proof. Let $\{G_i/i \in I\}$ be any τ_i open cover for X. since every τ_i open set is (i, j) semi open set, τ_i open cover will become (i, j) semi open cover and this contains finite (i, j) semi open cover. Since (X, τ_i, τ_j) is (i, j) semi compact. Therefore (X, τ_i) is compact. Similar argument gives the proof of the second part.

Corollary 3.6. If (X, τ_i, τ_j) be any (i, j) semi compact Bitopological space. Then (X, τ_i) is locally compact.

Theorem 3.7. A Bitopological space is (i, j) semi compact iff (i) Every class of (i, j) semi closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of (i, j) semi closed sets with finite intersection property has non empty intersection. (iii) iff Every basic (i, j) semi open cover has a finite sub cover.

Corollary 3.8. A Bitopological space is (i, j) semi compact iff (i) Every class of τ_i-closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of τ_i-closed sets with finite intersection property has non empty intersection. (iii) iff Every τ_i-basic open cover has a finite sub cover.

Theorem 3.9. Let $f\colon (X,\tau_i,\tau_j) \to (Y,\sigma_i,\sigma_j)$ be a pairwise semi continuous pairwise open map and (X,τ_i,τ_j) is (i,j) semi compact. Then $f(X)$ is (i,j) semi compact.

Proof. Let $\{G_i/i \in I\}$ be (i,j) semi open cover for f(X). Since f is pairwise semi continuous, $\{f^{-1}(G_i)/i \in I\}$ is (i,j) semi open cover for X and has a finite (i,j) semi open sub cover. Therefore, for f is pairwise open, $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f(X)$ is (i,j) semi compact.

Theorem 3.10. Let $f: (X,\tau_i,\tau_j) \to (Y,\sigma_i,\sigma_j)$ be a pairwise semi continuous pairwise open map and (Y,σ_i,σ_j) is (i,j) semi compact. Then $f^{-1}(Y)$ is (i,j)semi compact.

Proof. Let $\{G_i/i \in I\}$ be (i,j) semi open cover for $f^{-1}(Y)$ implies each G_i is (i,j) semi open set in $f^{-1}(Y)$, then there exists $U \in \tau_i$ such that $U \subset A \subset cl_j(U)$. Since f is pairwise semi continuous, we have $f(U) \subset f(A) \subset f(cl_j(U)) \subset cl_j(f(U))$. This shows that $f(G_i)$ is (i,j) semi open in Y and hence $\{f(G_i)/i \in I\}$ is (i,j) semi open cover for Y and has a finite (i,j) semi open sub cover. Since (Y,σ_i,σ_j) is (i,j) semi compact. Therefore $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f^{-1}(Y)$ is (i,j) semi compact.

Corollary 3.11. Let $f\colon (X,\tau_i,\tau_j) \to (Y,\sigma_i,\sigma_j)$ be pairwise continuous pairwise open map and (Y,σ_i,σ_j) is (i,j) semi compact. Then $f^{-1}(Y)$ is (i,j) semi compact.

Proof. since every pairwise continuous map is pairwise semi continuous, result follows directly from the above theorem.

Remark 2. [locally] (i,j) semi compact is weak hereditary.

§4. Pairwise semi compact Bitopological space

Definition 4.1. (X,τ_i,τ_j) is said to be (i) pairwise semi compact if it is (i,j) and (j,i) semi compact. (ii) locally pairwise semi compact if it is locally (i,j) semi compact and locally (j,i) semi compact.

Example 5. Let R be the space of reals with topologies τ_i and τ_j, where τ_i has $(a,b]$ as basic open sets, and τ_j has [a, b), $a < b$ as basic open sets, then (X,τ_i,τ_j) pairwise semi compact.

Example 6. since every open set is (i,j) semi open, (X,τ_i,τ_j), where $X = R, \tau_i = \{G_n/G_n = (-n,n)n \in Z^+\}$; $\tau_j = R_\Omega$ will act as the example for these two definitions.

Theorem 4.2. Every pairwise semi compact space is locally pairwise semi compact.

Proof. obvious from the definitions of pairwise semi compact and locally pairwise semi compact spaces.

Theorem 4.3. Let (X,τ_i,τ_j) be Bitopological space and $A \subseteq (X,\tau_i,\tau_j)$. Then A is pairwise semi compact subset of (X,τ_i,τ_j) iff the subspace $(A,\tau_{i/A},\tau_{j/A})$ is pairwise semi compact

Theorem 4.4. A Bi-closed subspace of a [locally] pairwise semi compact space is [locally] pairwise semi compact. The product of two [locally] pairwise semi compact spaces is [locally] pairwise semi compact.

Proof. $A \subseteq (X,\tau_i,\tau_j)$ is Bi-closed and (X,τ_i,τ_j) is pairwise semi compact. Let $U = \{U_i/i \in I\}, V = \{V_i/i \in I\}$ are respectively (i,j) and (j,i) semi open covers of A with respect to $\tau_{i/A}$ and $\tau_{j/A}$ then for each $U_i \in U$ and for each $V_i \in V$, fix an (i,j) semi open set G and an (j,i) semi open set H such that $U = G \cap A, V = H \cap A$, then the families $U' = \{G(U)/U \in$

$U\} \cup \{X - A\}$ and $V^{'} = \{H(V)/V \in V\} \cup \{X - A\}$ are (i,j) and (j, i) semi open covers of X with respect to τ_i and τ_j respectively, which implies $U^{'}$ and $V^{'}$ admits finite sub covers which in turn gives rise to the finite sub covers for U and V respectively. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi compact

Direct from Tychnoff's theorem.

Theorem 4.5. Let (X, τ_i, τ_j) be any [locally] pairwise semi compact Bitopological space. Then (X, τ_i) and (X, τ_j) are [locally] compact.

Corollary 4.6. If (X, τ_i, τ_j) be any pairwise semi compact Bitopological space. Then (X, τ_i) and (X, τ_j) are locally compact.

Theorem 4.7. A Bitopological space is Pairwise semi compact iff (i) Every class of pairwise closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of pairwise closed sets with finite intersection property has non empty intersection. (iii) iff Every pairwise basic open cover has a finite sub cover.

Theorem 4.8. A Bitopological space is Pairwise semi compact iff (i) Every class of Bi-closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of Bi-closed sets with finite intersection property has non empty intersection. (iii) iff Every pairwise basic open cover has a finite sub cover.

Theorem 4.9. Let $f\colon (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be a pairwise semi continuous pairwise open map and (X, τ_i, τ_j) is pairwise semi compact. Then $f(X)$ is pairwise semi compact.

Proof. Let $\{G_i/i \in I\}$ be pairwise semi open cover for $f(X)$. Then $\{G_i/i \in I\}$ is both (i, j) and (j, i) semi open covers for $f(X)$. since f is pairwise semi continuous, $\{f^{-1}(G_i)/i \in I\}$ is pairwise semi open cover for X and has a finite pairwise semi open sub cover. Therefore, for f is pairwise open, $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f(X)$ is pairwise semi compact.

Theorem 4.10. Let $f : (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be a pairwise semi continuous pairwise open map and (Y, σ_i, σ_j) is pairwise semi compact. Then $f^{-1}(Y)$ is pairwise semi compact.

Proof. Let $\{G_i/i \in I\}$ be pairwise semi open cover for $f^{-1}(Y)$ implies $\{f(G_i)/i \in I\}$ is pairwise semi open cover for Y and has a finite pairwise semi open sub cover. Since (Y, σ_i, σ_j) is pairwise semi compact. Therefore $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f^{-1}(Y)$ is pairwise semi compact.

Theorem 4.11. Let $f\colon (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ be pairwise continuous pairwise open map and (Y, σ_i, σ_j) is pairwise semi compact. Then $f^{-1}(Y)$ is pairwise semi compact.

Proof. since every pairwise continuous map is pairwise semi continuous, result follows directly from the above theorem.

Example 7. $X = R$; $\tau_i = \{\phi, R, (a, \infty) : a \in R\}$; $\tau_j = \{\phi, R, (-\infty, a) : a \in R\}$ then (X, τ_i, τ_j) is not (i,j) and pairwise semi compact. But according to Cooke and Reilly it is pairwise compact.

Example 8. $X = [0, 1]$; $\tau_i = \{\phi,\ X,\ \{0\}, [0, a) :\ a \in X\}$; $\tau_j = \{\phi,\ X,\ \{1\},\ (a, 1] : a \in X\}$ then $(X, \tau_i,\ \tau_j)$ is pairwise semi compact. According to Cooke and Reilly it is pairwise B-compact.

Example 9. $X = \{a, b, c\}$; $\tau_i = \{\phi,\ X,\ \{a, b\},\ \{c\}\}$; $\tau_j = \{\phi,\ X,\ \{a\},\ \{b, c\}\}$ then (X, τ_i, τ_j) is not B-compact according to Cooke and Reilly. Since {{a}, {b, c}} is a τ_j open

cover of X without τ_i open sub cover.

Example 10. $X = R^2$; τ_i = usual topology; τ_j = Half open rectangle topology; $A = \{(x,y)/0 \leq x < 1; 0 \leq y < 1\} \cup (1,1)$ is τ_j semi open w.r.to τ_i but neither τ_i open nor τ_j open.

Remark 3. [locally] pairwise semi compact is weak hereditary.

§5. (i,j) Semi second countability

Definition 5.1.

(i) (X, τ_i, τ_j) is said to be (i,j) semi second countable if it has countable (i,j) semi open base.

(ii) $(X,\ \tau_i,\ \tau_j)$ is said to be (i,j) semi first countable if each x in X has countable (i,j) semi open base.

Example 11. (X, τ_i, τ_j) is (i,j) semi second countable, where $X = R$, $\tau_i = R_\iota$; $\tau_j = \{G_n/G_n = (-n,n) : n \in N\}$.

Theorem 5.2.

(i) Any subspace of (i,j) semi [first] second countable space is (i,j) semi [first] second countable.

(ii) Any countable product of (i,j) semi [first] second countable spaces is (i,j) semi [first] second countable.

Proof. (i) Let $(A, \tau_{i/A}, \tau_{j/A})$ be any subspace of an (i,j) semi second countable space (X, τ_i, τ_j) and since (X, τ_i, τ_j) is (i,j) semi second countable, there exists countable (i,j) semi-open base $\{G_i\}$, which in turn gives $H_i = G_i \cap A$ a countable $(i_{/A}, j_{/A})$ semi open base for $(A, \tau_{i/A}, \tau_{j/A})$. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is (i,j) semi second countable. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is (i,j) semi first countable.

(ii) Let $X = \Pi X_i$; $X_i^{\prime}s$ are (i,j) semi second countable spaces and $B_i = \{B_i^j/j \in J\}$ is countable (i,j) semi open base for each X_i with respect to τ_1^i and τ_2^i respectively. Then $B = \Pi B_i = \{U_i/U_i = \Pi B_i^j$ where $B_i^j \in B_i$ for finite i and $B_j = X_j$ for $j \neq i\}$, is the countable (i,j) semi open base in (X, τ_1, τ_2) where $\tau_1 = \Pi_{i\in I}\tau_1^i$, $\tau_2 = \Pi_{i \in I}\tau_2^i$. Hence $X = \Pi X_i$ is (i,j) semi second countable. Similar argument shows $X = \Pi X_i$ is (i,j) semi first countable.

Corollary 5.3. Let (X, τ_i, τ_j) be any (i,j) semi [first] second countable Bitopological space. (i) Any subspace of (X, τ_i) [first] second countable space is (i,j) semi [first] second countable. (ii) Any countable product of (X, τ_i) [first] second countable spaces is (i,j) semi [first] second countable.

Remark 4. The property of (i,j) semi [first] second countability is hereditary.

§6. Pairwise semi second countability

Definition 6.1. (X, τ_i, τ_j) is said to be pairwise semi second [first] countable if it is both (i,j) and (j,i) semi second [first] countable.

Example 12. X = countable set; τ_i = cofinite topology and τ_j = discrete topology, is pairwise semi second countable.

Theorem 6.2. (i) Any subspace of pairwise semi [first] second countable space is pairwise semi [first] second countable. (ii) Any countable product of pairwise semi [first] second countable spaces is pairwise semi [first] second countable.

Proof. (i) Let $(A, \tau_{i/A}, \tau_{j/A})$ be any subspace of pairwise semi second countable space (X, τ_i, τ_j). $B = \{B_i/i \in I\}$, $B^{'} = \{B_i^{'}/i \in I\}$ are respectively countable (i, j) and (j, i) semi open bases in X. Then $C = \{C_i/C_i = B_i \cap A\}_{i \in I}$, $C^{'} = \{C_i^{'}/C_i^{'} = B_i^{'} \cap A\}_{i \in I}$ are countable $(i_{/A}, j_{/A})$ and $(j_{/A}, i_{/A})$ semi open bases for A. Therefore $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi second countable. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi first countable.

(ii) Let $X = \Pi X_i$; $X_i^{'}s$ are pairwise semi second countable, $B_i = \{B_i^j/j \in J\}$ and $B_i^{'} = \{B_{ij}^{'}/j \in J\}$ are countable (i, j) and (j, i) semi open bases for each X_i with respect to τ_1^i and τ_2^i respectively. Then $B = \Pi B_i = \{U_i/U_i = \Pi B_{ij}$ where $B_i^j \in B_i$ for finite i, and $B_j = X_j$ for $j \neq i\}$, $B^{'} = \Pi B_i^{'} = \{U_i^{'}/U_i^{'} = \Pi B_{ij}^{'}$ where $B_{ij}^{'} \in B_i^{'}$ for finite i, and $B_j^{'} = X_j$ for $j \neq i\}$ are respectively the countable(i, j) and (j, i) semi open bases in (X, τ_1, τ_2) where $\tau_1 = \Pi_{i \in I} \tau_1^i$, $\tau_2 = \Pi_{i \in I} \tau_2^i$. Hence $X = \Pi X_i$ is pairwise semi second countable. Similar argument shows $X = \Pi X_i$ is pairwise semi first countable.

Proofs of the following are trivial and so left to the reader.

Definition 6.3. (X, τ_i, τ_j) is said to be Bi second countable if it is both τ_i and τ_j second countable.

Corollary 6.4. Let (X, τ_i, τ_j) be any Bitopological space. (i) Every Bi [first] second countable Bitopological space is pairwise semi [first] second countable.

Corollary 6.5. Let (X, τ_i, τ_j) be any Bitopological space. (i) Any subspace of Bi first [second] countable space is pairwise semi first [second] countable. (ii) Any countable product of Bi first [second] countable space is pairwise semi first [second] countable.

Remark 5. The property of pairwise semi [first] second countability is hereditary.

§7. (i, j) semi Lindeloff Bitopological space

Definition 7.1. (X, τ_i, τ_j) is said to be (i, j) semi Lindeloff if each (i, j)semi open cover has a countable (i, j) semi open sub cover and (X, τ_i, τ_j) is locally (i, j) semi Lindeloff if each $x \in X$ has (i, j) semi Lindeloff neighborhood.

Example 13. Let X be the real line τ_i the lower limit topology and τ_j be the upper limit topology on X, then (X, τ_i, τ_j) is (i, j) semi Lindeloff.

Theorem 7.2. Suppose (X, τ_i, τ_j) has a countable (i, j) semi open base then every semi open cover of (X, τ_i, τ_j) contains a countable sub collection covering (X, τ_i, τ_j).

Proof. Let $B = \{B_n/n \in N\}$ is (i, j) countable semi open base for X. Let $U = \{U_i/i \in I\}$ be (i, j) semi open covering of X. For each $n \in N$, choose an element U_n of U such that $B_n \subseteq U_n$. Then the collection $U^{'} = \{U_n/n \in N\}$ is countable. Furthermore, it covers X. Given a point $x \in X$, we can choose an element U_n of U containing x. Since U_n is (i, j) semi open, there exists basis elements B_n, such that $x \in B_n \subseteq A$. Since $B_n \subseteq U_n$, it contains x. Therefore $x \in B_n \subseteq U_n$. Thus $U^{'}$ is a countable sub collection of U which covers X with respect to τ_i and τ_j respectively. Hence the theorem.

Theorem 7.3. (i) A τ_i-closed subspace of a [locally] (i, j) semi Lindeloff space is [locally] (i, j) semi Lindeloff. (ii) The product of two [locally] (i, j) semi Lindeloff spaces need not be [locally] (i, j) semi Lindeloff.

Proof. (i) Let A be τ_i-closed subspace of (i, j) semi Lindeloff space (X, τ_i, τ_j). Let $U =$

$\{U_i/i \in I\}$ be $(i_{/A}, j_{/A})$ semi open cover of A then for each $U_i \in U$ fix an (i,j) semi open set G such that $U = G \cap A$, then the family $U^{'} = \{G_i/i \in I\} \cup \{X - A\}$ is an (i,j) semi open cover of X, which implies $U^{'}$ admits countable sub cover which in turn gives countable $(i_{/A}, j_{/A})$ sub cover for A. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is (i,j) semi Lindeloff. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is locally (i,j) semi Lindeloff.

(ii) Remark 1 and standard theorem (sorgenfrey plane) gives the proof for this part.

Theorem 7.4. Every (i,j) semi [first] second countable space is [locally] (i,j) semi Lindeloff.

Corollary 7.5. Let (X, τ_i, τ_j) be any Bitopological space. If (X, τ_i) is [first] second countable then (X, τ_i, τ_j) is [locally] (i,j) semi Lindeloff.

Note. the converse of the above theorem is not true.

Theorem 7.6. If (X, τ_i, τ_j) is [locally] (i,j) semi compact Lindeloff Bitopological space, then (X, τ_i) is [locally] Lindeloff topological space.

Proof. Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i,j) semi open cover for X, then for (X, τ_i, τ_j) is (i,j) semi Lindeloff, $\{G_i/i \in I\}$ will have a countable sub cover which in turn becomes countable open cover for (X, τ_i). Hence (X, τ_i) is Lindeloff.

Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i,j) semi open cover for X, then for (X, τ_i, τ_j) is (i,j) semi compact, $\{G_i/i \in I\}$ will have a finite sub cover which in turn becomes finite open cover for (X, τ_i). Hence (X, τ_i) is Lindeloff. (since every compact space is Lindeloff).

Similar argument gives the proof of locally Lindeloffness.

Corollary 7.7. If (X, τ_i, τ_j) is (i,j) semi compact Lindeloff Bitopological space, then (X, τ_i) is locally Lindeloff topological space.

Theorem 7.8. If (X, τ_i, τ_j) is [locally] (i,j) semi Lindeloff and (Y, σ_i, σ_j) is [locally] (i,j) semi compact then $(X \times Y, \tau_i \times \sigma_i, \tau_j \times \sigma_j)$ is [locally] (i,j) semi Lindeloff.

Theorem 7.9. Every [locally] (i,j) semi compact space is [locally] (i,j) semi Lindeloff.

Remark 6. [locally] (i,j) semi Lindeloff is weak hereditary.

§8. Pairwise semi Lindeloff Bitopological space

Definition 8.1. (i) (X, τ_i, τ_j) is said to be pairwise semi Lindeloff if it is (i,j) and (j,i) semi Lindeloff and (ii) (X, τ_i, τ_j) is said to be locally pairwise semi Lindeloff if it is locally (i,j) semi Lindeloff and locally (j,i) semi Lindeloff.

Example 14. Let X be the real line τ_i the lower limit topology and τ_j be the upper limit topology on X, then (X, τ_i, τ_j) is pairwise semi Lindeloff.

Example 15. Let $X = R; \tau_i = R_\iota$ the lower limit topology and $\tau_j = R_\Omega$ where $R_\Omega =$ minimum well ordered uncountable set topology on X, then (X, τ_i, τ_j) is pairwise semi Lindeloff.

Theorem 8.2. Suppose (X, τ_i, τ_j) has a countable pairwise semi open base then every semi open cover of (X, τ_i, τ_j) contains a countable sub collection covering (X, τ_i, τ_j).

Proof. Let $B = \{B_n/n \in N\},\ B^{'} = \{B^{'}_n/n \in N\}$ are (i,j) and (j,i) countable semi open bases for X respectively. Let $U = \{U_i/i \in I\}, V = \{V_i/i \in I\}$ be (i,j) and (j,i) semi open coverings of X. For each $n \in N$, choose an element U_n of U; V_n of V such that $B_n \subseteq U_n$ and

$B_n^{'} \subseteq V_n$ respectively. Then the collection $U^{'} = \{U_n/n \in N\}$; $V^{'} = \{V_n/n \in N\}$ are countable. Furthermore, they covers X with respect to (i,j) and (j,i) respectively. Given a point $x \in X$, we can choose an element U_n of U containing x and V_n of V containing x. Since U_n, V_n are (i,j) and (j,i) semi open respectively, there exists basis elements B_n, $B_n^{'}$ such that $x \in B_n \subseteq A$ and $x \in B_n^{'} \subseteq A$. Since $B_n \subseteq U_n$, $B_n^{'} \subseteq V_n$ they both contains x. Therefore $x \in B_n \subseteq U_n$ and $x \in B_n^{'} \subseteq V_n$. Thus $U^{'}$ is a countable sub collection of U; $V^{'}$ is a countable sub collection of V, which covers X with respect to τ_1 and τ_2 respectively. Hence the theorem.

Definition 8.3. A subset in (X, τ_i, τ_j) is said to be a Bi-closed set if it is both τ_i and τ_j closed.

Theorem 8.4. (i) A Bi-closed subspace of a [locally] pairwise semi Lindeloff space is [locally] pairwise semi Lindeloff. (ii) The product of two [locally] pairwise semi Lindeloff spaces need not be [locally] pairwise semi Lindeloff.

Proof. (i) Let A be Bi-closed subspace of pairwise semi Lindeloff space (X, τ_i, τ_j). Let $U = \{U_i/i \in I\}, V = \{V_i/i \in I\}$ are respectively $(i_{/A}, j_{/A})$ and $(j_{/A}, i_{/A})$ semi open covers of A then for each $U_i \in U$ and for each $V_i \in V$, fix an (i,j) semi open set G and (j,i) semi open set H such that $U = G \cap A, V = H \cap A$, then the families $U^{'} = \{G_i/i \in I\} \cup \{X - A\}$ and $V^{'} = \{H_i/i \in I\} \cup \{X - A\}$ are (i,j) and (j,i) semi open covers of X respectively, which implies $U^{'}$ and $V^{'}$ admits countable sub covers which in turn gives countable sub covers for A. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi Lindeloff. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is locally pairwise semi Lindeloff.

(ii) Let (X, τ_i, τ_j) be pairwise semi Lindeloff space, where $X = R,\ \tau_i = R_\iota,\ \tau_j = R_\upsilon$ then $X \times X$ is the product of two pairwise semi Lindeloff spaces called sorgenfrey planes. It has basis all sets of the form $[a,b)X[c,d)$ and $(b,-a]X(d,c]$ in the planes. Consider the subspaces $L = \{xX(-x)/x \in R_\iota\}$ and $L^{'} = \{(-x)Xx/x \in R_\upsilon\}$ in R_ι^2 and R_υ^2 respectively. Then obviously L and $L^{'}$ are (i,j) and (j,i) semi closed in R_ι^2 and R_υ^2 respectively. By covering R^2 by the (i,j) semi open sets $R_\iota^2 - L$ and by the basis elements of the form $[a,b)X[-a,d)$ and by (j,i) semi open sets $R_\upsilon^2 - L^{'}$ and by the basis elements of the form $(b,-a]X[d,a)$ we can see that each of these basis elements in R_ι^2 and R_υ^2 intersects L and $L^{'}$ in at most one point respectively. Since both L and $L^{'}$ are uncountable, no countable sub collection covers R_ι^2 and R_υ^2 respectively. Thus the product of two pairwise semi Lindeloff Bitopological spaces is not pairwise semi Lindeloff.

Theorem 8.5. Every Bi - [first] second countable space is [locally] pairwise semi Lindeloff.

Theorem 8.6. If (X, τ_i, τ_j) is [locally] pairwise semi compact Lindeloff Bitopological space, then (X, τ_i) and (X, τ_j) are [locally] Lindeloff topological spaces

Proof. Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i,j) semi open cover for X, then for (X, τ_i, τ_j) is (i,j) semi Lindeloff, $\{G_i/i \in I\}$ will have a countable sub cover which in turn becomes countable open cover for (X, τ_i). Hence (X, τ_i) is Lindeloff.

Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i,j) semi open cover for X, then for (X, τ_i, τ_j) is (i,j) semi compact, $\{G_i/i \in I\}$ will have a finite sub cover which in turn becomes finite open cover for (X, τ_i). Hence (X, τ_i) is Lindeloff(since every compact space is Lindeloff).

Similar argument gives the proof of locally Lindeloffness.

Corollary 8.7. If (X, τ_i, τ_j) is pairwise semi (compact)Lindeloff Bitopological space, then

(X, τ_i) and (X, τ_j) are locally Lindeloff topological space.

Theorem 8.8. If (X, τ_i, τ_j) is [locally] pairwise semi Lindeloff and (Y, σ_i, σ_j) is [locally] pairwise semi compact then $(X \times Y, \tau_i \times \sigma_i, \tau_j \times \sigma_j)$ is [locally] pairwise semi Lindeloff.

Corollary 8.9. If (X, τ_i, τ_j) is pairwise semi Lindeloff and (Y, σ_i, σ_j) is pairwise semi compact then $(X \times Y, \tau_i \times \sigma_i, \tau_j \times \sigma_j)$ is pairwise locally semi Lindeloff.

Theorem 8.10. Every [locally] pairwise semi compact space is [locally] pairwise semi Lindeloff.

Proof. Let (X, τ_i, τ_j) is pairwise semi compact. Then for G_i an (i, j) and $G_i^{'}$ an (j, i) semi open covers with finite (i, j) and (j, i) semi open covers for X respectively. Since every finite cover is countable cover, G_i an (i, j) and $G_i^{'}$ an (j, i) are semi open covers with countable (i, j) and (j, i) semi open sub covers for X respectively. Hence (X, τ_i, τ_j) is pairwise semi Lindeloff. Trivial verification shows (X, τ_i, τ_j) is [locally] pairwise semi Lindeloff.

Remark 7. [locally] pairwise semi Lindeloff is weak hereditary.

Conclusion. In this paper we defined new compactness and Lindeloffness in Bitopological spaces and studied their interrelations.

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A note on f-minimum functions

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Abstract For a given arithmetical function $f : \mathbb{N} \to \mathbb{N}$, let $F : \mathbb{N} \to \mathbb{N}$ be defined by $F(n) = \min\{m \geq 1 : n|f(m)\}$, if this exists. Such functions, introduced in [4], will be called as the f-minimum functions. If f satisfies the property $a \leq b \Longrightarrow f(a)|f(b)$, we shall prove that $F(ab) = \max\{F(a), F(b)\}$ for $(a, b) = 1$. For a more restrictive class of functions, we will determine $F(n)$ where n is an even perfect number. These results are generalizations of theorems from [10], [1], [3], [6].

Keywords Divisibility of integers, prime factorization, arithmetical functions, perfect numbers.

§1. Introduction

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers, and $f : \mathbb{N} \to \mathbb{N}$ a given arithmetical function, such that for each $n \in \mathbb{N}$ there exists at least an $m \in \mathbb{N}$ such that $n|f(m)$. In 1999 and 2000 [4], [5], as a common generalization of many arithmetical functions, we have defined the application $F : \mathbb{N} \to \mathbb{N}$ given by

$$F(n) = \min\{m \geq 1 : n|f(m)\}, \tag{1}$$

called as the "f-minimum function". Particularly, for $f(m) = m!$ one obtains the Smarandache function (see [10], [1])

$$S(n) = \min\{m \geq 1 : n|m!\}. \tag{2}$$

Moree and Roskam [2], and independently the author [4], [5], have considered the Euler minimum function

$$E(n) = \min\{m \geq 1 : n|\varphi(n)\}, \tag{3}$$

where φ is Euler's totient. Many other particular cases of (1), as well as, their "dual" or analogues functions have been studied in the literature; for a survey of concepts and results, see [9].

In 1980 Smarandache discovered the following basic property of $S(n)$ given by (2):

$$S(ab) = \max\{S(a), S(b)\} \text{ for } (a, b) = 1. \tag{4}$$

Our aim in what follows is to extend property (4) to a general class of f-minimum functions. Further, for a subclass we will be able to determine $F(n)$ for even perfect numbers n.

§2. Main results

Theorem 1. Suppose that F of (1) is well defined. Then for distinct primes p_i, and arbitrary $\alpha_i \geq 1$ $(i = 1, 2 \cdots, r)$ one has

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \geq \max\{F(p_i^{\alpha_i}) : i = 1, 2 \cdots, r\}. \tag{5}$$

The second result offers a reverse inequality:

Theorem 2. With the notations of Theorem 1 suppose that f satisfies the following divisibility condition:

$$a|b \Longrightarrow f(a)|f(b) \quad (a, b \geq 1) \tag{$*$}$$

Then one has

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq l.c.m.\{F(p_i^{\alpha_i}) : i = 1, 2 \cdots, r\}, \tag{6}$$

where $l.c.m.$ denotes the least common multiple.

By replacing $(*)$ with another condition, a more precise result is obtainable:

Theorem 3. Suppose that f satisfies the condition:

$$a \leq b \Longrightarrow f(a)|f(b) \;\; (a, b \geq 1). \tag{$**$}$$

Then

$$F(mn) = \max\{F(m), F(n)\} \text{ for } (m, n) = 1. \tag{7}$$

Finally, we shall prove the following:

Theorem 4. Suppose that f satisfies $(**)$ and the following two assumptions:

(i) $n|f(n)$; (ii) For each prime p and $m < p$ we have $p \nmid f(n)$. (8)

Let k be an even perfect number. Then

$$F(k) = k/2^s, \text{ where } 2^s \| k. \tag{9}$$

Remarks . (1) The function φ satisfies property $(*)$. Then relation (6) gives a result for the Euler minimum function $E(n)$ (see [7], [8]).

(2) Let $f(m) = m!$. Then clearly $(**)$ holds true. Thus (7) extends relation (4). For another example, let $f(m) = l.c.m.\{1, 2, \ldots, m\}$. Then the function F given by (1) satisfies again (7), proved e.g. in [1].

(3) If $f(n) = n!$, then both (i) and (ii) of (8) are satisfied. This relation (9) for $F \equiv S$ follows. This was first proved in [3] (see also [6]).

§3. Proof of theorems

Theorem 1. There is no loss of generality to prove (5) for $r = 2$. Let p^α, q^β be two distinct prime powers. Then

$$F(p^\alpha q^\beta) = \min\{n \geq 1 : p^\alpha q^\beta | f(m)\} = m_0,$$

so $p^\alpha q^\beta | f(m_0)$. This is equivalent to $p^\alpha | f(m_0)$, $q^\beta | f(m_0)$. By definition (1) we get $m_0 \geq F(p^\alpha)$ and $m_0 \geq F(q^\beta)$, i.e. $F(p^\alpha q^\beta) \geq \max\{F(p^\alpha), F(q^\beta)\}$. It is immediate that the same proof applies to $F\left(\prod p^\alpha\right) \geq \max\{F(p^\alpha)\}$, where p^α are distinct prime powers.

Theorem 2. Let $F(p^\alpha) = m_1$, $F(q^\beta) = m_2$. By definition (1) of function F it follows that $p^\alpha | F(m_1)$ and $q^\beta | F(m_2)$. Let $l.c.m.\{m_1, m_2\} = g$. Since $m_1 | g$, one has $f(m_1) | f(g)$ by $(*)$. Similarly, since $m_2 | g$, one can write $f(m_2) | f(g)$. These imply $p^\alpha | f(m_1) | f(g)$ and $q^\beta | f(m_2) | f(g)$, yielding $p^\alpha q^\beta | f(g)$. By definition (1) this gives $g \geq F(p^\alpha q^\beta)$, i.e. $l.c.m.\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta)$, proving the theorem for $r = 2$. The general case follows exactly by the same lines.

Theorem 3. By taking into account of (5), one needs only to show that the reverse inequality is true. For simplicity, let us consider again $r = 2$. Let $F(p^\alpha) = m$, $F(q^\beta) = n$ with $m \leq n$. By definition (1) one has $p^\alpha | f(m)$, $q^\beta | f(n)$. Now, by assumption $(**)$ we can write $f(m) | f(n)$, so $p^\alpha | f(m) | f(n)$. Therefore, one has $p^\alpha | f(n)$, $q^\beta | f(n)$. This in turn implies $p^\alpha q^\beta | f(n)$, so $n \geq F(p^\alpha q^\beta)$; i.e. $\max\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta)$. The general case follows exactly the same lines. Thus, we have proved essentially, that $F(p^\alpha q^\beta) = \max\{F(p^\alpha), F(q^\beta)\}$, or more generally

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) = \max\{F(p_i^{\alpha_i}) : i = 1, 2 \cdots, r\}. \tag{10}$$

Now, relation (7) is an immediate consequence of (10), for by writing

$$m = \prod_{i=1}^{r} p_i^{\alpha_i}, \quad n = \prod_{j=1}^{s} q_j^{\beta_j}, \text{ with } (p_i, q_j) = 1,$$

it follows that

$$F(mn) = \max\{F(p_i^{\alpha_i}), F(q_j^{\beta_j}) : i = 1, 2 \cdots, r, j = 1, 2 \cdots, s\}$$
$$= \max\{\max\{E(p_i^{\alpha_i}) : i = 1, 2 \cdots, r\}, \max\{E(q_j^{\beta_j}) : j = 1, 2 \cdots, s\}\}$$
$$= \max\{F(m), F(n)\},$$

by equality (10).

Theorem 4. By (i) and definition (1) we get

$$F(n) \leq n. \tag{11}$$

Now, by (i), one has $p | f(p)$ for any prime p, but by (ii), p is the least such number. This implies that

$$F(p) = p \text{ for any prime } p. \tag{12}$$

Now, let k be an even perfect number. By the Euclid-Euler theorem (see e.g. [7]) k may be written as $k = 2^{n-1}(2^n - 1)$, where $p = 2^n - 1$ is a prime ("Mersenne prime"). Since $(**)$ holds true, by Theorem 3 we can write

$$F(k) = F(2^{n-1}(2^n - 1)) = \max\{F(2^{n-1}), F(2^n - 1)\}.$$

Since $F(2^n - 1) = 2^n - 1$ (by (12)), and $F(2^{n-1}) \leq 2^{n-1}$ (by (11)), from $2^{n-1} < 2^n - 1$ for $n \geq 2$, we get $F(k) = 2^n - 1 = \dfrac{k}{2^s}$, where $s = n - 1$ and $2^s \| k$. This finishes the proof of Theorem 4.

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Some Smarandache conclusions of coloring properties of complete uniform mixed hypergraphs deleted some $\mathcal{C}$-hyperedges

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Abstract The upper and lower chromatic number of uniform mixed hypergraphs and $\mathcal{C}$-hyperedge and $\mathcal{D}$-hyperedge contact with the inevitable. In general, the increase in the $\mathcal{C}$-hyperedge will increase lower chromatic number $\chi_{\mathcal{H}}$, an increase $\mathcal{D}$-hyperedge will decrease upper chromatic number $\overline{\chi}_{\mathcal{H}}$. In this paper the relationship between $\mathcal{C}$-hyperedge with the upper chromatic number and lower chromatic number and some conclusions with respect to mixed hypergraph are given.

Keywords A complete uniform mixed hypergraph, chromatic, upper chromatic, Smarandache conclusions.

§1. Lemma and the basic concepts

Definition 1.1.[1] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, $\mathcal{C} = \{C_1, C_2, \cdots, C_l\}$, $\mathcal{D} = \{D_1, D_2, \cdots, D_m\}$ are two subset clusters of X, all of which $C_i \in \mathcal{C}$ to meet $|C_i| \geq 2$, and all $D_j \in \mathcal{D}$ to meet $|D_j| \geq 2$. Then $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called a mixed hypergraph from X, and each $C_i \in \mathcal{C}$ is called the $\mathcal{C}-$hyperedges, and each $D_j \in \mathcal{D}$ is called the $\mathcal{D}-$hyperedges. In particular, $\mathcal{H}_D = (X, \mathcal{D})$ is called a $\mathcal{D}$ hypergraph, the $\mathcal{H}_C = (X, \mathcal{C})$ for $\mathcal{C}-$ hypergraph.

Definition 1.2.[2] On $2 \leq l, m \leq n = |X|$, let

$$\mathcal{K}(n, l, m) = (X, \mathcal{C}, \mathcal{D}) = (X, \binom{X}{l}, \binom{X}{m})$$

where $|\mathcal{C}| = \binom{n}{l}$ and $|\mathcal{D}| = \binom{n}{m}$, then $\mathcal{K}(n, l, m)$ is called the complete (l, m)-uniform mixed hypergraph with n vertex.

It is clear that for a given n, l, m, in a sense of the isomorphic existence just has one $\mathcal{K}(n, l, m)$.

Definition 1.3.[3,4] For mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, the largest i among all existence of strict i-Coloring known as the upper chromatic number $\mathcal{H}$, said that for $\bar{\chi}_{\mathcal{H}}$.

Definition 1.4.[5] For mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if a i Partition $X = \{X_1, X_2, \cdots, X_i\}$ of vertex sets X satisfy:

1) For each $\mathcal{C}$-hyperedge at least two vertices is allocated in the same block;
2) For each $\mathcal{D}$-hyperedge at least two vertices is allocated in different blocks.

The partition is called as a feasible partition of $\mathcal{H}$.

Obviously, any strict i coloring of $\mathcal{H}$ corresponds with a strict i feasible partition, and vice versa. they are equivalent. Therefore, we write one feasible partition of $\mathcal{H}$ or a strict i-coloring c as: $c = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_i$ and $r_i(\mathcal{H}) = r_i$ is the total number of all feasible i partition.

Definition 1.5.[6] Let S be a subset of the vertices set X of mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if the set does not contain any of the $\mathcal{C}$-Hyperedge ($\mathcal{D}$-Hyperedge) as a subset, then it is called $\mathcal{C}$ stable or $\mathcal{C}$ independent ($\mathcal{D}$ stable or $\mathcal{D}$ independent).

Lemma 1.1.[7,8] Let mixed hypergraph $\mathcal{H} = (X, \binom{X}{r}, \mathcal{D})$, where $2 \le r \le n = n(\mathcal{H})$, then Arbitrary a coloring of $\mathcal{H}$ meet condition

$$\overline{\chi}(\mathcal{H}) = r - 1.$$

Definition 1.6.[9] mixed hypergraph for $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if there is a mapping $c : Y \to \{1, 2, \cdots, \lambda\}$ that between subset $Y \in X$ and λ colors $\{1, 2, \cdots, \lambda\}$, and it meet following conditions:

1)For each $\mathcal{C}$-hyperedge $C \in \mathcal{C}$, at least two vertices are the same color;

2) For each $\mathcal{D}$-hyperedge $D \in \mathcal{D}$, at least two vertices are different colors.

Then, mapping c is called as one λ colors normal coloring of the mixed hypergraph $\mathcal{H}$.

Definition 1.7.[10] In a normal $i-$coloring of $\mathcal{H}$, if i colors are used, then the coloring is called a strict i-coloring.

It is clear that a normal $\chi(\mathcal{H})-$coloring of mixed hypergraph $\mathcal{H}$ must be a strict coloring.

Definition 1.8.[11] For any coloring c of the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, Let Y be a subset of X, then if Y satisfied: the arbitrary $y_1 \in Y, y_2 \in Y$, there is $c(y_1) = c(y_2)$, then we call the subset Y as monochrome; if each of two is different colors, that is, $c(y_1) \neq c(y_2)$, then we call subset Y as the multi-color.

By the definitionthe of the normal coloring of mixed hypergraph, we know that for any normal coloring of the hypergraph, $\mathcal{D}$-hyperedge not is a subset of monochrome, $\mathcal{C}$-hyperedge not is a subset of the multi-color.

Definition 1.9.[12] In an arbitrary strictly i-coloring of $\mathcal{H}$, the vertex‘s set X of $\mathcal{H}$ certainly is divided into i partition, each partition is a non-empty subset of monochrome, we call it as the color category.

Lemma 1.2.[13] Let mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, and $n = |X|$, then regardless $\mathcal{H}$ can normal coloring or not can, but the coloring of its sub-hypergraph $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{D}}$ is always available and there is $\chi(\mathcal{H}_{\mathcal{C}}) = 1, r_1(\mathcal{H}_{\mathcal{C}}) = 1$ and $\overline{\chi}(\mathcal{H}_{\mathcal{D}}) = n(\mathcal{H}), r_n(\mathcal{H}_{\mathcal{D}}) = 1$.

Lemma 1.3.[14] For mixed hypergraph $\mathcal{H}' = (X, \mathcal{C}, \binom{X}{m})$, if $\forall C \in \mathcal{C}$, where $|C| = k$ and $n(\mathcal{H}') \le (k-1)(m-1)$, then $\overline{\chi}(\mathcal{H}') \ge k - 1$.

Lemma 1.4. Mixed hypergraph for $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, let $\mathcal{H}'$ be arbitrary a subhypergraph of $\mathcal{H}$, the $\chi(\mathcal{H}) \ge \chi(\mathcal{H}')$, $\bar{\chi}(\mathcal{H}) \le \bar{\chi}(\mathcal{H}')$.

Lemma 1.5.[15] For a colorable mixed hypergraph $\mathcal{H} = (X, \binom{X}{l}, \binom{X}{m})$, where $2 \le l, m \le n$, then

1) $\chi(\mathcal{H}) = \lceil \frac{n(\mathcal{H})}{m-1} \rceil$, $\overline{\chi}(\mathcal{H}) = l - 1$;

2) $\mathcal{H}$ is the uncolorable if and only if $\lceil \frac{n(\mathcal{H})}{m-1} \rceil \ge l$.

Lemma 1.6.[16] For $\mathcal{C}$ hypergraph $\mathcal{H} = (X, \mathcal{C}, \emptyset)$, if $\forall C \in \mathcal{C}$ there are $|C| \ge k$, then $\overline{\chi}(\mathcal{H}) \ge k - 1$, and all $(k-1)$-coloring of $\mathcal{H}$ are normal.

Lemma 1.7.[17] If m articles put into n box, then at least one box contain not less than $\left\lceil \frac{m}{n} \right\rceil$ articles, where $\left\lceil \frac{m}{n} \right\rceil$ is not less than $\frac{m}{n}$ smallest integer.

Lemma 1.8.[18] $\mathcal{H}=(X,\mathcal{C},\mathcal{D})$ be anyone of the normal coloring of the mixed hypergraph $\mathcal{H}$, then

$$1 \le \chi(\mathcal{H}_{\mathcal{D}}) \le \chi(\mathcal{H}) \le \overline{\chi}(\mathcal{H}) \le \overline{\chi}(\mathcal{H}_{\mathcal{C}}) \le n.$$

Lemma 1.9.[19] For the mixed hypergraph $\mathcal{H}^{'}=(X,\binom{X}{l}-\mathcal{C}^{'},\binom{X}{m})$, where $\mathcal{C}^{'}\subseteq\binom{X}{l}$, and $n=|X|$, then

1) When $n\le(l-1)(m-1)$, $\mathcal{H}^{'}$ certainly at least has a normal $(l-1)$ coloring, that is, $\mathcal{H}^{'}$ certainly is colorable.

2) If $\mathcal{H}^{'}$ is the colorable, then $n\ge(l-1)(m-1)+1$.

§2. The main results

Theorem. Let $\mathcal{H}^{'}=(X,\binom{X}{l}-\mathcal{C}^{'},\binom{X}{m})$ where $\mathcal{C}^{'}\subseteq\binom{X}{l}$, and $|\mathcal{C}^{'}|=k$, then

1) When $k=1$, there are

I) If $|X|=l$, then $\overline{\chi}(\mathcal{H}^{'})=l$;

II) If $l+1\le|X|\le(l-1)(m-1)$, then $\overline{\chi}(\mathcal{H}^{'})=l-1$.

2) When $k=2$, there are

I) If $|X|=l+1$, then $\overline{\chi}(\mathcal{H}^{'})=l$;

II) If $l+2\le|X|\le(l-1)(m-1)$, then $\overline{\chi}(\mathcal{H}^{'})=l-1$.

3) When $k\ge3$, if $l+k-1\le|X|\le(l-1)(m-1)$, then $\overline{\chi}(\mathcal{H}^{'})=l-1$.

Proof. 1) When $k=1$, it is necessary to $|\binom{X}{l}|\ge1$, so $|X|\ge l$. There are two cases as follows:

I) If $|X|=l$, then $\mathcal{H}^{'}$ does not contain $\mathcal{C}$-hyperedge. We can see the $\overline{\chi}(\mathcal{H}^{'})=n(\mathcal{H}^{'})=l$ from Lemma 1.2;

II) If $l+1\le|X|\le(l-1)(m-1)$, By Lemma 1.6, $\overline{\chi}(\mathcal{H}^{'})\ge l-1$.

As long as prove $\overline{\chi}(\mathcal{H}^{'})\le l-1$, that's all.

Use reduction to absurdity. If not, that is $\overline{\chi}(\mathcal{H}^{'})\ge l$. there are two cases.

2.1. If $\overline{\chi}(\mathcal{H}^{'})\ge l+1$, then there is a the feasible $\overline{\chi}(\mathcal{H}^{'})$ partition of $\mathcal{H}^{'}$, Let $X=X_1\bigcup X_2\bigcup\cdots\bigcup X_{\overline{\chi}(\mathcal{H}^{'})}$, where each $X_i(i=1,2,\cdots,\overline{\chi}(\mathcal{H}^{'}))$ non-empty, And $X_i\bigcap X_j=\emptyset(i\ne j)$, then there is $\overline{\chi}(\mathcal{H}^{'})$ separate vertex $x_i\in X_i(i=1,2,\cdots,\overline{\chi}(\mathcal{H}^{'})$, makes $\{x_1,x_2,\cdots,x_{\overline{\chi}(\mathcal{H}^{'})}\}$ for a $\mathcal{C}$-stability set. Thus $\overline{\chi}(\mathcal{H}^{'})$ vertex does not form any $\mathcal{C}$-hyperedge, so, $|\mathcal{C}^{'}|\ge\binom{\overline{\chi}(\mathcal{H}^{'})}{l}$.

We also available from the assumption that: $|\mathcal{C}^{'}|\ge\binom{l+1}{l}=l+1>2$, this is contradictory with $|\mathcal{C}^{'}|=1$. So, $\overline{\chi}(\mathcal{H}^{'})\ge l+1$ is impossible.

2.2. If $\overline{\chi}(\mathcal{H}^{'})=l$, then there is a feasible l partition of $\mathcal{H}^{'}$, let $X=X_1\bigcup X_2\bigcup\cdots\bigcup X_l$, as a result of $l+1\le|X|\le(l-1)(m-1)$, therefore, by the lemma 1.7, at least a color category contains at least $\lceil\frac{|X|}{l}\rceil\ge\lceil\frac{l+1}{l}\rceil=2$ vertices. assume it is X_1. As the others of the color classes are non-empty, so certainly there are $l-1$ different points: $x_i\in X_i(i=2,3,\cdots,l)$, they together with $x_1,x_1^{'}\in X_1$ formate two $\mathcal{C}$-stability sets $\{x_1,x_2,\cdots,x_l\}$ and $\{x_1^{'},x_2,\cdots,x_l\}$ For $\mathcal{H}^{'}$. That is, $\mathcal{H}^{'}$ at least does not contain them. So, $|\mathcal{C}^{'}|\ge2$. This with the known condition $|\mathcal{C}^{'}|=1$ contradictions. So $\overline{\chi}(\mathcal{H}^{'})\ne l$. By case 2.1 and 2.2, we can see that $\overline{\chi}(\mathcal{H}^{'})\le l-1$.

Overview on, we can see that when $k=1$, if $l+1\le|X|\le(l-1)(m-1)$, then $\overline{\chi}(\mathcal{H}^{'})=l-1$.

2) When $k=2$, it is necessary to $|\binom{X}{l}|\ge2$, so $|X|\ge l+1$. There are two cases.

Case 1. If $|X| = l+1$, Let $X = \{x_1, x_2, \cdots, x_l, x_{l+1}\}$, then two $\mathcal{C}$-hyperedge in $\mathcal{C}'$ certainly have $l-1$ vertices are the same, that is, and there is only one Vertex is not the same. Let they are $C_1 = \{x_1, x_2, \cdots, x_l\}$ and $C_2 = \{x_2, \cdots, x_{l+1}\}$. Use the following coloring method for $\mathcal{H}'$ coloring: Allocate color 1 for vertex x_1 and x_{l+1}, the remaining $l-1$ vertices $x_2, \cdots, x_l$ separately allocate colors $2, 3, \cdots, l$, clearly, this is a strict l coloring of $\mathcal{H}'$, therefore $\overline{\chi}(\mathcal{H}') \geq l$.

Following prove that $\overline{\chi}(\mathcal{H}') \leq l$.

Use reduction to absurdity. If not, then $\overline{\chi}(\mathcal{H}') \geq l+1$. For easily state, we let $\overline{\chi}(\mathcal{H}') = k$. as a result of $|X| = l+1$, Therefore, by the lemma 1.7 and lemma 1.8 we can see $k \leq l+1$, then only is $k = l+1$. As a result, certainly, there is a feasible $l+1$ Partition of $\mathcal{H}'$, let it is $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_{l+1}$, where $X_i \neq \emptyset (i = 1, 2, \cdots, l+1)$, and $X_i \bigcap X_j = \emptyset (i \neq j)$. Thus, there is $l+1$ different vertices $x_i \in X_i (i = 1, 2, \cdots, l+1)$, they formate set $\{x_1, x_2, \cdots, x_{l+1}\}$, clearly, this is a $\mathcal{C}$-stability set of $\mathcal{H}'$. That is, $\{x_1, x_2, \cdots, x_{l+1}\}$ does not contain any $\mathcal{C}$-hyperedge, then, $|\mathcal{C}'| \geq \binom{l+1}{l} = l+1 > 2$, This is in contradiction with the known conditions. Therefore, $k \neq l+1$. So that $\overline{\chi}(\mathcal{H}') \leq l$.

Overview above, we can see that when $k = 2$, if $|X| = l+1$, then $\overline{\chi}(\mathcal{H}') = l$.

Cases 2. If $l+2 \leq |X| \leq (l-1)(m-1)$, we can see from Lemma 1.3 $\overline{\chi}(\mathcal{H}') \geq l-1$. As long as proved that $\overline{\chi}(\mathcal{H}') \leq l-1$, that's all.

If not, then $\overline{\chi}(\mathcal{H}') \geq l$. For the convenience of expression, we let that $\overline{\chi}(\mathcal{H}') = s$, from the assumption we easy know that $s \geq l$. there are two circumstances under which says:

Case 2.1. If $s = l$, by the definition of the strict normal coloring and the feasible partition, we can see that certainly there is at least one the feasible l Partition of $\mathcal{H}'$, let it is $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_l$, where $X_i \neq \emptyset (i = 1, 2, \cdots, l)$ and $X_i \bigcap X_j = \emptyset (i \neq j)$.

Because $l+2 \leq |X| \leq (l-1)(m-1)$, therefore, the partition $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_l$ may be the only the following two circumstances:

i) At least one color class include at least with more than 3 vertices, and the remaining color classes include at least a vertex;

ii) At least two color classes include at least two or more vertices, and the remaining color classes include at least a vertex.

When the case i), we let color class X_1 include at least more than 3 vertices and the rest include at least a vertex, and let $x_1, x_1', x_1'' \in X_1$, $x_2 \in X_2, \cdots, x_l \in X_l$. by the definition of feasible partition, we know that $\{x_1, x_2, \cdots, x_l\}$, $\{x_1', x_2, \cdots, x_l\}$ and $\{x_1'', x_2, \cdots, x_l\}$ are three $\mathcal{C}$-stability Sets of mixed hypergraph $\mathcal{H}'$. Then $\mathcal{H}'$ at least does not contain they. that is, they are included by $\mathcal{C}'$, So, $|\mathcal{C}'| \geq 3$, This contradictions with known condition $|\mathcal{C}'| = 2$, so $s \neq l$.

When the case ii), we let that color class X_1 and X_2 contain at least two or more vertices, and the remaining color classes contain at least a vertex, Let $x_1, x_1' \in X_1, x_2, x_2' \in X_2$, $x_3 \in X_3, \cdots, x_l \in X_l$. by the definition of the feasible partition, we know that $\{x_1, x_2, \cdots, x_l\}$, $\{x_1, x_2', \cdots, x_l\}$, $\{x_1', x_2, \cdots, x_l\}$ and $\{x_1', x_2', \cdots, x_l\}$ are 4 $\mathcal{C}$-stability set of the mixed hypergraph $\mathcal{H}'$. That is, $\mathcal{H}'$ at least does not contain them. So, $|\mathcal{C}'| \geq 4$, This contradictions with known condition $|\mathcal{C}'| = 2$. so $s \neq l$under this circumstances.

Overview above, we know that if $l+2 \leq |X| \leq (l-1)(m-1)$, then $s \neq l$.

Case 2.2. If $s \geq l+1$, by the definition of the strict normal coloring and the feasible parti-

tion, we know that certainly there is at least one the feasible s Partition $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_s$ of $\mathcal{H}^{'}$, where $X_i \neq \emptyset (i = 1, 2, \cdots, s)$ and $X_i \bigcap X_j = \emptyset (i \neq j)$, then exist s different from each other vertices $x_i \in X_i (i = 1, 2, \cdots, s)$, makes $\{x_1, x_2, \cdots, x_s\}$ for the $\mathcal{C}$-stability set of $\mathcal{H}^{'}$. That is, $\{x_1, x_2, \cdots, x_s\}$ dose not contain any $\mathcal{C}$-hyperedge, so $|\mathcal{C}^{'}| \geq \binom{s}{l} \geq \binom{l+1}{l} = l + 1 > 2$, this contradictions with known conditions $|\mathcal{C}^{'}| = k = 2$. So, assumption condition $s \geq l + 1$ does not hold.

Integrated proven of cases 2.1 and cases 2.2 we know that $\overline{\chi}(\mathcal{H}^{'}) \leq l - 1$.

As a result, when $k = 2$, if $l + 2 \leq |X| \leq (l - 1)(m - 1)$, then $\overline{\chi}(\mathcal{H}^{'}) = l - 1$.

Through the proved course of case 1 and case 2, we know that the conclusion of the (2) is proper.

3) When $k \geq 3$, if $l + k - 1 \leq |X| \leq (l - 1)(m - 1)$, by the conclusion 1) of a lemma 1.9, we know that $\mathcal{H}^{'}$ is colorable, and by Lemma 1.3 we know that the $\overline{\chi}(\mathcal{H}^{'}) \geq l - 1$.

We prove $\overline{\chi}(\mathcal{H}^{'}) \leq l - 1$, that's all.

Still we use the reduction to absurdity. If not, assume that $\overline{\chi}(\mathcal{H}^{'}) \geq l$. For described be convenient, we let $\overline{\chi}(\mathcal{H}^{'}) = r$.

Through the definition of the Strictly normal coloring and the feasible partition, we know that there is a certain, the feasible r Partition of $\mathcal{H}^{'}$: $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_r$, where $X_i \neq \emptyset (i = 1, 2, \cdots, r)$, $X_i \bigcap X_j = \emptyset (i \neq j)$.

As each color classes is non-empty, so there is $x_i \in X_i (i = 1, 2, \cdots, r)$, and because $|X| \geq l + k - 1$,

Therefore, in addition to the X contain $x_1, x_2, \cdots, x_r$ these r vertices, It also includes at least $l + k - 1 - r$ vertices: $x_1^{'}, x_2^{'}, \cdots, x_{(l+k-1-r)}^{'}$. The $l + k - 1 - r$ vertices arbitrarily are assigned to above r color classes, possible different assigned methods only are the following cases:

Case 1. Above $l + k - 1 - r$ vertices were separately assigned to the $l + k - 1 - r$ the color classes;

Case 2. They were assigned to a total of $l + k - 2 - r$ the color classes;

...

Case $l + k - 1 - r$. They were assigned to the same color class.

Before study these cases, we prove the following assertion:

Assertion. Let $s = |\{Y | Y \text{is } \mathcal{C}\text{-stability Set of mixed hypergraph } \mathcal{H}^{'}, \text{and} |Y| \geq l\}|$, in above all the $l + k - 1 - r$ different cases, the $l + k - 1 - r$ kind of cases correspond s value is the minimum among all.

Proof. To make the following agreement: For any the feasible r Partition $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_r$ of the $\mathcal{H}^{'}$, We assume $|X_1| \geq |X_2| \geq \cdots \geq |X_r|$. Obviously, the s value only is related with the vertex's number of each color class, and is not related with the order of all color classes arrayed. Thus under the condition of keep the s value uncharge, through exchange color classes order, we can do this. In this agreement, According to above the $l + k - 1 - r$ kind of all circumstances Distributed $x_1^{'}, x_2^{'}, \cdots, x_{(l+k-1-r)}^{'}$ to each color class,then the feasible r Partition of correspondence with this kind distributed certainly meet to the following condition:

$|X_1| \geq l + k - r, |X_i| \geq 1 (i = 2, 3, \cdots, r)$, we may let $|X_1| = n_1, |X_2| = n_2, \cdots, |X_r| = n_r$, then the s value of corresponding to this feasible partition is:

$$\begin{aligned}
s_0 &= \sum_{i=l}^{r} \sum_{\substack{\{j_1,j_2,\cdots,j_i\}\subseteq\{1,2,\cdots,r\}\\ |X_{j_1}|\geq|X_{j_2}|\geq\cdots\geq|X_{j_i}|}} |X_{j_1}||X_{j_2}|\cdots|X_{j_i}| \\
&= \sum_{i=l}^{r}[\ |X_1| \sum_{\substack{\{j_1,j_2,\cdots,j_{i-1}\}\subseteq\{2,\cdots,r\}\\ |X_{j_1}|\geq|X_{j_2}|\geq\cdots\geq|X_{j_{i-1}}|}} |X_{j_1}||X_{j_2}|\cdots|X_{j_{i-1}}| \\
&\quad + \sum_{\substack{\{j_1^{'},j_2^{'},\cdots,j_i^{'}\}\subseteq\{2,\cdots,r\}\\ |X_{j_1^{'}}|\geq|X_{j_2^{'}}|\geq\cdots\geq|X_{j_i^{'}}|}} |X_{j_1^{'}}||X_{j_2^{'}}|\cdots|X_{j_i^{'}}|\ ] \\
&\geq (l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-1} + \sum_{i=l}^{r}\binom{r-1}{i}. \qquad (1)
\end{aligned}$$

It is easy to know that $(l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-1}+\sum_{i=l}^{r}\binom{r-1}{i}$ precisely represent to the s value corresponding with the feasible partition $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_r$, where $x_i \in X_i (i = 1, 2, \cdots, r)$, $x_j^{'} \in X_1 (j = 1, 2, \cdots, l+k-1-r)$, this the feasible partition is attained through allocate $x_1, x_2, \cdots, x_r, x_1^{'}, x_2^{'}, \cdots, x_{(l+k-1-r)}^{'}$ to each color class according to above the $l+k-1-r$ kinds of cases,and when $X = \{x_1, x_2, \cdots, x_r x_1^{'}, x_2^{'}, \cdots, x_{(l+k-1-r)}^{'}\}$, That is, when $n(\mathcal{H}^{'}) = l+k-1$.

Combination formula (1), we know that the s value when $n(\mathcal{H}^{'}) > l+k-1$ is greater than when $n(\mathcal{H}^{'}) = l+k-1$. thus, in order to make access to the minimum s, have to be $|X| = n(\mathcal{H}^{'}) = l+k-1$.

The following as further prove that when $|X| = l+k-1$, only according to above the $l+k-1-r$ kind of cases allocate vertices to each color class, corresponding s value is the minimum.

In accordance with the above-mentioned in article, we let $X = \{x_1, x_2, \cdots, x_r, x_1^{'}, x_2^{'}, \cdots, x_{(l+k-1-r)}^{'}\}$, and let $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_r$, where $x_i \in X_i (i = 1, 2, \cdots, r), x_j^{'} \in X_1 (j = 1, 2, \cdots, l+k-1-r)$, is a feasible partition corresponding with above the $l+k-1-r$ kind of cases. then other cases corresponding to the feasible partition can be seen as is according to following method obtained:

Selecting out some vertices from color class X_1, then put them in other color classes, and keep the relationship $|X_{j_1}| \geq |X_{j_2}| \geq \cdots \geq |X_{j_i}|$.

Let the vertices number which selected out from X_1 for p, following through mathematical induction on p to prove that assertion.

When $p = 1$, we may let selected out from X_1 vertex for $x_{(l+k-1-r)}^{'}$, by above the agreement, we know that the feasible partition of re-distributed the vertex $x_{(l+k-1-r)}^{'}$ only may is: $X_1 = \{x_1, x_1^{'}, x_2^{'}, \cdots, x_{(l+k-2-r)}^{'}\}$, $X_2 = \{x_2, x_{(l+k-2-r)}^{'}\}$, $X_i = \{x_i\} (i = 3, \cdots, r)$, clearly, corresponding with it s value is:

$$\begin{aligned}
&\sum_{i=l}^{r} \sum_{\substack{\{j_1,j_2,\cdots,j_i\}\subseteq\{1,2,\cdots,r\}\\ |X_{j_1}|\geq|X_{j_2}|\geq\cdots\geq|X_{j_i}|}} |X_{j_1}||X_{j_2}|\cdots|X_{j_i}| \\
&= |X_1|\sum_{i=l}^{r}\binom{r-2}{i-1} + |X_2|\sum_{i=l}^{r}\binom{r-2}{i-1} + |X_1||X_2|\sum_{i=l}^{r}\binom{r-2}{i-2} + \sum_{i=l}^{r}\binom{r-2}{i} \\
&= [(l+k-r)+2]\sum_{i=l}^{r}\binom{r-2}{i-1} + 2(l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-2} + \sum_{i=l}^{r}\binom{r-2}{i}.
\end{aligned}$$

In addition, it is clear that the $l+k-1-r$ kinds of cases corresponding to the feasible Partition $X = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_r$, where $x_i \in X_i (i = 1, 2, \cdots, r), x_j^{'} \in X_1 (j = 1, 2, \cdots, l+k-1-r)$ corresponding s value is:

$$\sum_{i=l}^{r} \sum_{\substack{\{j_1, j_2, \cdots, j_i\} \subseteq \{1,2,\cdots,r\} \\ |X_{j_1}| \geq |X_{j_2}| \geq \cdots \geq |X_{j_i}|}} |X_{j_1}||X_{j_2}| \cdots |X_{j_i}|$$

$$= |X_1| \sum_{i=l}^{r} \tbinom{r-2}{i-1} + \sum_{i=l}^{r} \tbinom{r-1}{i}$$

$$= (l+k-r) \sum_{i=l}^{r} \tbinom{r-2}{i-1} + \sum_{i=l}^{r} \tbinom{r-1}{i}.$$

It is clear that

$$[(l+k-r)+2] \sum_{i=l}^{r} \binom{r-2}{i-1} \geq (l+k-r) \sum_{i=l}^{r} \binom{r-2}{i-1},$$

$$2(l+k-r) \sum_{i=l}^{r} \binom{r-2}{i-2} + \sum_{i=l}^{r} \binom{r-2}{i} \geq \sum_{i=l}^{r} \binom{r-1}{i},$$

As a result,

$$[(l+k-r)+2] \sum_{i=l}^{r} \binom{r-2}{i-1} + 2(l+k-r) \sum_{i=l}^{r} \binom{r-2}{i-2} + \sum_{i=l}^{r} \binom{r-2}{i}$$

$$\geq (l+k-r) \sum_{i=l}^{r} \binom{r-2}{i-1} + \sum_{i=l}^{r} \binom{r-1}{i}.$$

That is, when $p = 1$, the conclusion is true.

Assumption that when $p < t$ the conclusions also are true. Then when $p = t$, that is, from the color class X_1 select out t vertices re-assigned to other color classes and to maintain relations $|X_{j_1}| \geq |X_{j_2}| \geq \cdots \geq |X_{j_i}|$.

We may let that the t time selected out from X_1 vertex is $x_{(l+k-t-r)}^{'}$, and it will be re-assigned to the color class X_j. And let this color classes $X_2, \cdots, X_{j-1}, X_j, X_{j+1}, \cdots, X_r$ contained in the vertex total number different were: $n_2, \cdots, n_{j-1}, n_j, n_{j+1}, \cdots, n_r$, at the same time we let that this step operation before the color class X_1 contained Vertices number for $l+k-r-t+1$, then after the operation of this step each color class contained vertex number different are $|X_1| = l+k-r-t$, $|X_2| = n_2, \cdots, |X_{j-1}| = n_{j-1}, |X_j| = n_j + 1, |X_{j+1}| = n_{j+1}, \cdots, |X_r| = n_r$.

It is easy to know s value corresponding with the feasible partition that obtained by after the $t-1$ step operation is:

$$
\begin{aligned}
s_{t-1} &= \sum_{i=l}^{r} \sum_{\substack{\{j_1,j_2,\cdots,j_i\}\subseteq\{1,2,\cdots,r\}\\ |X_{j_1}|\geq|X_{j_2}|\geq\cdots\geq|X_{j_i}|}} |X_{j_1}||X_{j_2}|\cdots|X_{j_i}| \\
&= \sum_{i=l}^{r} \Bigg[(l+k-t+1) \sum_{\substack{\{j_1,j_2,\cdots,j_{i-1}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1<j_2<\cdots<j_{i-1}}} n_{j_1}n_{j_2}\cdots n_{j_{i-1}} \\
&\quad + n_j \sum_{\substack{\{j_1',j_2',\cdots,j_{i-1}'\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1'<j_2'<\cdots<j_{i-1}'}} n_{j_1'}n_{j_2'}\cdots n_{j_{i-1}'} \\
&\quad + (l+k-t+1)n_j \sum_{\substack{\{j_1'',j_2'',\cdots,j_{i-2}''\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1''<j_2''<\cdots<j_{i-2}''}} n_{j_1''}n_{j_2''}\cdots n_{j_{i-2}''} \\
&\quad + \sum_{\substack{\{j_1''',j_2''',\cdots,j_i'''\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1'''<j_2'''<\cdots<j_i'''}} n_{j_1'''}n_{j_2'''}\cdots n_{j_i'''} \Bigg] \\
&= \sum_{i=l}^{r} \Bigg[(l+k-t+n_j+1) \sum_{\substack{\{j_1,j_2,\cdots,j_{i-1}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1<j_2<\cdots<j_{i-1}}} n_{j_1}n_{j_2}\cdots n_{j_{i-1}} \\
&\quad + (l+k-t+1)n_j \sum_{\substack{\{j_1'',j_2'',\cdots,j_{i-2}''\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1''<j_2''<\cdots<j_{i-2}''}} n_{j_1''}n_{j_2''}\cdots n_{j_{i-2}''} \\
&\quad + \sum_{\substack{\{j_1''',j_2''',\cdots,j_i'''\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1'''<j_2'''<\cdots<j_i'''}} n_{j_1'''}n_{j_2'''}\cdots n_{j_i'''} \Bigg].
\end{aligned}
$$

By the assumption of mathematical induction we know that $s_{t-1} \geq (l+k-r)\sum\limits_{i=l}^{r}\binom{r-2}{i-1} + \sum\limits_{i=l}^{r}\binom{r-1}{i}$

Therefore, the s value corresponding to the feasible partition of that after the T step operation is:

$$
\begin{aligned}
s_t &= \sum_{i=l}^{r} \sum_{\substack{\{j_1,j_2,\cdots,j_i\}\subseteq\{1,2,\cdots,r\}\\ |X_{j_1}|\geq|X_{j_2}|\geq\cdots\geq|X_{j_i}|}} |X_{j_1}||X_{j_2}|\cdots|X_{j_i}| \\
&= \sum_{i=l}^{r} \Bigg[(l+k-t) \sum_{\substack{\{j_1,j_2,\cdots,j_{i-1}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1<j_2<\cdots<j_{i-1}}} n_{j_1}n_{j_2}\cdots n_{j_{i-1}} \\
&\quad +(n_j+1) \sum_{\substack{\{j'_1,j'_2,\cdots,j'_{i-1}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j'_1<j'_2<\cdots<j'_{i-1}}} n_{j'_1}n_{j'_2}\cdots n_{j'_{i-1}} \\
&\quad +(l+k-t)(n_j+1) \sum_{\substack{\{j''_1,j''_2,\cdots,j''_{i-2}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j''_1<j''_2<\cdots<j''_{i-2}}} n_{j''_1}n_{j''_2}\cdots n_{j''_{i-2}} \\
&\quad + \sum_{\substack{\{j'''_1,j'''_2,\cdots,j'''_i\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j'''_1<j'''_2<\cdots<j'''_i}} n_{j'''_1}n_{j'''_2}\cdots n_{j'''_i} \Bigg] \\
&= \sum_{i=l}^{r} \Bigg[(l+k-t+n_j+1) \sum_{\substack{\{j_1,j_2,\cdots,j_{i-1}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j_1<j_2<\cdots<j_{i-1}}} n_{j_1}n_{j_2}\cdots n_{j_{i-1}} \\
&\quad +(l+k-t+1)n_j \sum_{\substack{\{j''_1,j''_2,\cdots,j''_{i-2}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j''_1<j''_2<\cdots<j''_{i-2}}} n_{j''_1}n_{j''_2}\cdots n_{j''_{i-2}} \\
&\quad + \sum_{\substack{\{j'''_1,j'''_2,\cdots,j'''_i\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j'''_1<j'''_2<\cdots<j'''_i}} n_{j'''_1}n_{j'''_2}\cdots n_{j'''_i} \Bigg] \\
&\quad +(l+k-t-n_j) \sum_{\substack{\{j''_1,j''_2,\cdots,j''_{i-2}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j''_1<j''_2<\cdots<j''_{i-2}}} n_{j''_1}n_{j''_2}\cdots n_{j''_{i-2}} \\
&= s_{t-1} + (l+k-t-n_j) \sum_{\substack{\{j''_1,j''_2,\cdots,j''_{i-2}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j''_1<j''_2<\cdots<j''_{i-2}}} n_{j''_1}n_{j''_2}\cdots n_{j''_{i-2}} \\
&\geq (l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-1} + \sum_{i=l}^{r}\binom{r-1}{i} \\
&\quad +(l+k-t-n_j) \sum_{\substack{\{j''_1,j''_2,\cdots,j''_{i-2}\}\subseteq\{2,\cdots,j-1,j+1,\cdots,r\}\\ j''_1<j''_2<\cdots<j''_{i-2}}} n_{j''_1}n_{j''_2}\cdots n_{j''_{i-2}}.
\end{aligned}
$$

According to the agreement given when begin prove,after t step operation,the vertices' number which contained in color class X_1 is $l+k-r-t$, and it is not less than that any other color classes contained, namely $l+k-r-t \geq n_j+1$. Then $l+k-t-n_j \geq t+1 > 0$. Therefore,

the feasible partition of that obtained after the t step operation corresponding value of s is

$$s_t \geq (l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-1}+\sum_{i=l}^{r}\binom{r-1}{i}.$$

That is, when $p=t$, the conclusions are true.

Overview above, we know that the conclusions are true for any natural number p.

In the following we prove this theorem conclusions on the basis of above assertion, It is clear that by the assertion we the easy access to the following conclusions:

1) When $\overline{\chi}(\mathcal{H}^{'})=l$, by the assertion we know that when the $n(\mathcal{H}^{'})=l+k-1$, s value is the minimum and $s_{min}=k$; if the vertices number $n(\mathcal{H}^{'})>l+k-1$, then $s>k$.

2) When $n(\mathcal{H}^{'})\geq l+k-1$ and it is a definite value, and $\overline{\chi}(\mathcal{H}^{'})>l$, by above assertion, We know that the value of s corresponding to this condition certainly greater than k.

Combination of above two conclusions, we know that when $l+k-1\leq |X|\leq (l-1)(m-1)$, $\overline{\chi}(\mathcal{H}^{'})\geq l$ must have $s>k$.

Again we return to the theorem proof. By the assumption $\overline{\chi}(\mathcal{H}^{'})\geq l$ of that given when begin prove conclusions from 3) and the known conditions $l+k-1<|X|\leq (l-1)(m-1)$, and combination proved just conclusion, we know $s>k$.

Because s represents the total number of $\mathcal{C}$-stable set that is contained from one the feasible partition corresponding to a strict $\overline{\chi}(\mathcal{H}^{'})\geq l$ coloring of $\mathcal{H}^{'}$,and the cardinal number of $\mathcal{C}$-stable set is greater than l or equal to l, Because, each $\mathcal{C}$-stable set which the cardinal number of set is greater than l or equal to l contain at least one $\mathcal{C}$-hyperedge of the mixed hypergraph $\mathcal{H}=(X,\binom{X}{l},\binom{X}{m})$. so mixed hypergraph $\mathcal{H}^{'}$ should does not contain at least s $\mathcal{C}$-hyperedge, that is, $|\mathcal{C}^{'}|\geq s>k$, with known conditions $|\mathcal{C}^{'}|=k$ contradictions. Then, the assume is untrue. Therefore, $\overline{\chi}(\mathcal{H}^{'})\leq l-1$.

By above the whole process of prove, we know that when $k\geq 3$, if $l+k-1<|X|\leq (l-1)(m-1)$, then $\overline{\chi}(\mathcal{H}^{'})=l-1$.

Corollary. Let $\mathcal{H}^{'}=(X,\binom{X}{l}-\mathcal{C}^{'},\binom{X}{m})$, where $\mathcal{C}^{'}\subseteq\binom{X}{l}$, and let $|\mathcal{C}^{'}|=k$, and s be the number of $\mathcal{C}$-stable set of $\mathcal{H}^{'}$, then

$$s_{min}=(l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-1}+\sum_{i=l}^{r}\binom{r-1}{i}+\sum_{i=1}^{l-1}\binom{n}{i}$$

where $n=|X|$.

Proof. $\mathcal{C}$-stable set of $\mathcal{H}^{'}$ is divided into two groups: one is those the cardinal number less than l; the other is those the cardinal number not less than l. By the above assertion and its proof we know that the total number of $\mathcal{C}$-stable set those cardinal number not less than l at least equal to $(l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-1}+\sum_{i=l}^{r}\binom{r-1}{i}$. It is easy obtain that total number of $\mathcal{C}$-stable set those the cardinal number less than l is $\sum_{i=1}^{l-1}\binom{n}{i}$, so,

$$s_{min}=(l+k-r)\sum_{i=l}^{r}\binom{r-2}{i-1}+\sum_{i=l}^{r}\binom{r-1}{i}+\sum_{i=1}^{l-1}\binom{n}{i}.$$

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A limit problem of the Smarandache dual function $S^{**}(n)$[1]

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Abstract For any positive integer n, the Smarandache dual function $S^{**}(n)$ is defined as

$$S^{**}(n) = \begin{cases} \max\{2m: \ m \in N^*, \ (2m)!! \mid n\}, & 2 \mid n; \\ \max\{2m-1: \ m \in N^*, \ (2m-1)!! \mid n\}, & 2 \nmid n. \end{cases}$$

The main purpose of this paper is using the elementary methods to study the convergent properties of an infinity series involving $S^{**}(n)$, and give an interesting limit formula for it.

Keywords The Smarandache dual function, limit problem, elementary method.

§1. Introduction and Results

For any positive integer n, the Smarandache dual function $S^{**}(n)$ is defined as the greatest positive integer $2m-1$ such that $(2m-1)!!$ divide n, if n is an odd number; $S^{**}(n)$ is the greatest positive $2m$ such that $(2m)!!$ divides n, if n is an even number. From the definition of $S^{**}(n)$ we know that the first few values of $S^{**}(n)$ are: $S^{**}(1) = 1$, $S^{**}(2) = 2$, $S^{**}(3) = 3$, $S^{**}(4) = 2$, $S^{**}(5) = 1$, $S^{**}(6) = 2$, $S^{**}(7) = 1$, $S^{**}(8) = 4$, $\cdots$. About the elementary properties of $S^{**}(2)$, some authors had studied it, and obtained many interesting results. For example, Su Gou [1] proved that for any real number $s > 1$, the series $\sum\limits_{n=1}^{\infty} \frac{S^{**}(n)}{n^s}$ is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s} = \zeta(s)\left(1 - \frac{1}{2^s}\right)\left(1 + \sum_{m=1}^{\infty} \frac{2}{((2m+1)!!)^s}\right) + \zeta(s)\left(\sum_{m=1}^{\infty} \frac{2}{((2m)!!)^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Yanting Yang [2] studied the mean value estimate of $S^{**}(n)$, and gave an interesting asymptotic formula:

$$\sum_{n \le x} S^{**}(n) = x\left(2e^{\frac{1}{2}} - 3 + 2e^{\frac{1}{2}} \int_0^1 e^{-\frac{y^2}{2}} dy\right) + O(\ln^2 x),$$

where $e = 2.7182818284\cdots$ is a constant.

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Yang Wang [3] also studied the mean value properties of $S^{**}(n)^2$, and prove that

$$\sum_{n\leq x} S^{**}(n)^2 = \frac{13x}{2} + O\left(\left(\frac{\ln x}{\ln\ln x}\right)^3\right).$$

In this paper, we using the elementary method to study the convergent properties of the series

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s},$$

and give an interesting identity and limit theorem. That is, we shall prove the following:
Theorem. For any real number $s > 1$, we have the identity

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} = \zeta(s)\left[1 - \frac{1}{2^s} + \left(1 - \frac{1}{2^s}\right)\sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s} + \sum_{m=1}^{\infty} \frac{8m-4}{((2m)!!)^s}\right],$$

where $\zeta(s)$ is the Riemann zeta-function.

From this Theorem we may immediately deduce the following limit formula:

Corollary. We have the limit

$$\lim_{s\to 1}(s-1)\left(\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s}\right) = \frac{13}{2}.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. It is clear that $S^{**}(n) \ll \ln n$, so if $s > 1$, then the series $\sum\limits_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s}$ is convergent absolutely, so we have

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} = \sum_{\substack{n=1\\ 2\nmid n}}^{\infty} \frac{S^{**}(n)^2}{n^s} + \sum_{\substack{n=1\\ 2\mid n}}^{\infty} \frac{S^{**}(n)^2}{n^s} \equiv S_1 + S_2,$$

where

$$S_1 = \sum_{\substack{n=1\\ 2\nmid n}}^{\infty} \frac{S^{**}(n)^2}{n^s}, \quad S_2 = \sum_{\substack{n=1\\ 2\mid n}}^{\infty} \frac{S^{**}(n)^2}{n^s}.$$

From the definition of $S^{**}(n)$ we know that if $2 \nmid n$, we can assume that $S^{**}(n) = 2m-1$, then $(2m-1)!! \mid n$. Let $n = (2m-1)!!u$, $2m+1 \nmid u$. Note that the identity

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \left(1 - \frac{1}{2^s}\right)\sum_{n=1}^{\infty} \frac{1}{n^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s),$$

so from the definition of $S^{**}(n)$ we can deduce that ($s>1$),

$$\begin{aligned}
S_1 &= \sum_{m=1}^{\infty}\sum_{\substack{u=1,\ 2\nmid u\\ 2m+1\nmid u}}^{\infty}\frac{(2m-1)^2}{((2m-1)!!)^s\,u^s}\\
&= \sum_{m=1}^{\infty}\frac{(2m-1)^2}{((2m-1)!!)^s}\sum_{\substack{u=1,\ 2\nmid u\\ 2m+1\nmid u}}^{\infty}\frac{1}{u^s}\\
&= \sum_{m=1}^{\infty}\frac{(2m-1)^2}{((2m-1)!!)^s}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}-\frac{1}{(2m+1)^s}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}\right)\\
&= \zeta(s)\left(1-\frac{1}{2^s}\right)\left(\sum_{m=1}^{\infty}\frac{(2m-1)^2}{((2m-1)!!)^s}-\sum_{m=1}^{\infty}\frac{(2m-1)^2}{((2m+1)!!)^s}\right)\\
&= \zeta(s)\left(1-\frac{1}{2^s}\right)\left(1+\sum_{m=1}^{\infty}\frac{(2m+1)^2-(2m-1)^2}{((2m+1)!!)^s}\right)\\
&= \zeta(s)\left(1-\frac{1}{2^s}\right)\left(1+\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}\right).
\end{aligned}$$

For even number n, we assume that $S^{**}(n)=2m$, then $(2m)!!\mid n$. Let $n=(2m)!!v$, $2m+2\nmid v$. If $s>1$, then we can deduce that

$$\begin{aligned}
S_2 &= \sum_{m=1}^{\infty}\sum_{\substack{v=1\\ 2m+2\nmid v}}^{\infty}\frac{(2m)^2}{((2m)!!)^s v^s}\\
&= \sum_{m=1}^{\infty}\frac{(2m)^2}{((2m)!!)^s}\sum_{\substack{v=1\\ (2m+2)\nmid v}}^{\infty}\frac{1}{v^s}\\
&= \sum_{m=1}^{\infty}\frac{(2m)^2}{((2m)!!)^s}\left(\sum_{n=1}^{\infty}\frac{1}{n^s}-\frac{1}{(2m+2)^s}\sum_{n=1}^{\infty}\frac{1}{n^s}\right)\\
&= \zeta(s)\left(\sum_{m=1}^{\infty}\frac{(2m)^2}{((2m)!!)^s}-\sum_{m=1}^{\infty}\frac{(2m)^2}{((2m+2)!!)^s}\right)\\
&= \zeta(s)\left(\frac{1}{2^{s-2}}+\sum_{m=1}^{\infty}\frac{(2m+2)^2-(2m)^2}{((2m+2)!!)^s}\right)\\
&= \zeta(s)\left(\frac{1}{2^{s-2}}+\sum_{m=1}^{\infty}\frac{8m+4}{((2m+2)!!)^s}\right)\\
&= 4\zeta(s)\sum_{m=1}^{\infty}\frac{2m-1}{((2m)!!)^s}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{S^{**}(n)^2}{n^s} = S_1 + S_2\\
= &\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1+\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}\right)+4\zeta(s)\sum_{m=1}^{\infty}\frac{2m-1}{((2m)!!)^s}\\
= &\zeta(s)\left[1-\frac{1}{2^s}+\left(1-\frac{1}{2^s}\right)\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}+\sum_{m=1}^{\infty}\frac{8m-4}{((2m)!!)^s}\right].
\end{aligned}$$

This completes the proof of our Theorem.

Now we prove Corollary, note that

$$\begin{aligned}
&\frac{1}{2}+\sum_{m=1}^{\infty}\frac{4m}{(2m+1)!!}+\sum_{m=1}^{\infty}\frac{8m-4}{(2m)!!}\\
= &\frac{1}{2}+\sum_{m=1}^{\infty}\left(\frac{2}{(2m-1)!!}-\frac{2}{(2m+1)!!}\right)+\sum_{m=1}^{\infty}\left(\frac{4}{(2m-2)!!}-\frac{4}{(2m+2)!!}\right)\\
= &\frac{1}{2}+2+4=\frac{13}{2}
\end{aligned}$$

and

$$\lim_{s\to 1}(s-1)\zeta(s)=1,$$

from Theorem we may immediately deduce that

$$\begin{aligned}
&\lim_{s\to 1}(s-1)\left(\sum_{n=1}^{\infty}\frac{S^{**}(n)}{n^s}\right)\\
= &\lim_{s\to 1}(s-1)\zeta(s)\left[1-\frac{1}{2^s}+\left(1-\frac{1}{2^s}\right)\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}+\sum_{m=1}^{\infty}\frac{8m-4}{((2m)!!)^s}\right]\\
= &\frac{1}{2}+\sum_{m=1}^{\infty}\frac{4m}{(2m+1)!!}+\sum_{m=1}^{\infty}\frac{8m-4}{(2m)!!}=\frac{13}{2}.
\end{aligned}$$

This completes the proof of Corollary.

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Separation axioms in smooth fuzzy topological spaces

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Abstract In this paper, the concepts of fuzzy $\widetilde{g} - T_i$ $(i = 0, 1, 2, 1/2)$ spaces, fuzzy $\widetilde{g}$-normal spaces and fuzzy $\widetilde{g}$-regular spaces in the sense of Sostak [7] are introduced. Also, some interesting properties and characterizations of them are investigated.

Keywords Fuzzy $\widetilde{g} - Ti$ $(i = 0, 1, 2, 1/2)$ spaces, fuzzy $\widetilde{g}$-normal spaces, fuzzy $\widetilde{g}$-regular spaces.

§1. Introduction and Preliminaries

The concept of fuzzy set was introduced by Zadeh [10] in his classical paper. Fuzzy sets have applications in many fields such as information [6] and control [8]. In 1985, Sostak [7] introduced a new form of topological structure. In 1992, Ramadan [4] studied the concept of smooth fuzzy topological spaces. The concept of $\widetilde{g}$-open set was discussed by Rajesh and Erdal Ekici [3]. The concept of fuzzy normal spaces was introduced by Bruce Hutton [1]. Kubiak [9] established many interesting properties of fuzzy normal spaces. The purpose of this paper is to introduce fuzzy $\widetilde{g} - Ti$ $(i = 0, 1, 2, 1/2)$ spaces, $\widetilde{g}$-normality and $\widetilde{g}$-regularity in smooth fuzzy topological spaces. Also many interesting characterizations are established.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\langle \in I$, $T(x) = \langle$ for all x $\in$ X.

Definition 1.1. [2] A fuzzy point x_t in X is a fuzzy set taking value $t \in I_0$ at x and zero elsewhere, $x_t \in \lambda$ iff $t \leq \lambda(x)$. A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by λ q μ, if there exists x $\in$ X such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \ \overline{q} \ \mu$.

Definition 1.2. [7] A function $T : I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions :

(1) $T(\overline{0}) = T(\overline{1}) = 1$;

(2) $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.

(3) $T(\underset{j \in \Gamma}{\vee} \mu_j) \geq \underset{j \in \Gamma}{\wedge} T(\mu_j)$ for any $\{ \mu_j \}_{j \in \Gamma} \in$ I^{X}.

The pair (X, T) is called a smooth fuzzy topological space.

Remark 1.1. Let (X, T) be a smooth fuzzy topological space. Then, for each $r \in I_0, T_r = \{\mu \in I^X : T(\mu) \geq r\}$ is Chang's fuzzy topology on X.

Definition 1.3. [5] Let (X, T) be a smooth fuzzy topological space. For each $\lambda \in I^X, r \in I_0$, an operator $C_T : I^X \times I_0 \to I^X$ is defined as follows: $C_T(\lambda, r) = \wedge\{\mu : \mu \geq \lambda, T(\overline{1} - \mu) \geq r\}$.

For λ, $\mu \in I^X$ and r, s $\in I_0$, it satisfies the following conditions:
(1) $C_T(\overline{0}, r) = \overline{0}$.
(2) $\lambda \leq C_T(\lambda, r)$.
(3) $C_T(\lambda, r) \vee C_T(\mu, r) = C_T(\lambda \vee \mu, r)$.
(4) $C_T(\lambda, r) \leq C_T(\lambda, s)$, if $r \leq s$.
(5) $C_T(C_T(\lambda, r), r) = C_T(\lambda, r)$.

Proposition 1.1. [5] Let (X, T) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $I_T : I^X \times I_0 \to I^X$ is defined as follows: $I_T(\lambda, r) = \vee\{\mu : \mu \leq \lambda, T(\mu) \geq r\}$. For λ, $\mu \in I^X$ and r, s $\in I_0$, it satisfies the following conditions:
(1) $I_T(\overline{1} - \lambda, r) = \overline{1} - C_T(\lambda, r)$.
(2) $I_T(\overline{1}, r) = \overline{1}$.
(3) $\lambda \geq I_T(\lambda, r)$.
(4) $I_T(\lambda, r) \wedge I_T(\mu, r) = I_T(\lambda \wedge \mu, r)$.
(5) $I_T(\lambda, r) \geq I_T(\lambda, s), \mathrm{if} r \leq s$.
(6) $I_T(I_T(\lambda, r), r) = I_T(\lambda, r)$.

Definition 1.4. [4] Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,
(1) $SC_T(\lambda, r) = \wedge\{\mu \in I^X : \mu \geq \lambda, \mu \text{ is r} - \text{fuzzy semiclosed}\}$ is called r-fuzzy semiclosure of λ.
(2) λ is called r-fuzzy semiclosed (briefly, r-fsc) if $\lambda \geq I_T(C_T(\lambda, r), r)$.
(3) λ is called r-fuzzy semiopen (briefly, r-fso) if $\lambda \leq C_T(I_T(\lambda, r), r)$.

Definition 1.5. [4] Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be a function. Then
(1) f is called fuzzy continuous iff $S(\mu) \leq T(f^{-1}(\mu))$ for each $\mu \in I^Y$.
(2) f is called fuzzy open iff $T(\lambda) \leq S(f(\lambda))$ for each $\lambda \in I^X$.

§2. Fuzzy $\widetilde{g}$ $-T_i$ spaces

In this section, the concept of fuzzy $\widetilde{g}$ $-T_i$ $(i = 0, 1, 2, 1/2)$ spaces is introduced. Interesting properties and characterizations of such spaces are discussed.

Definition 2.1. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called
(1) r-fuzzy $\hat{g}$-closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-fuzzy semiopen. The complement of a r-fuzzy $\hat{g}$-closed set is said to be a r-fuzzy $\hat{g}$-open set.
(2) r-fuzzy *g-closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-fuzzy $\hat{g}$-open. The complement of a r-fuzzy *g-closed set is said to be a r-fuzzy *g-open set.
(3) r-fuzzy #g-semiclosed (briefly r-#fgs-closed) if $SC_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-fuzzy *g-open. The complement of a r-fuzzy #g-semiclosed set is said to be a r-fuzzy #g-semiopen set (briefly r-#fgs-open set).
(4) r-fuzzy $\widetilde{g}$-closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-#fgs-open. The complement of a r-fuzzy $\widetilde{g}$-closed set is said to be a r-fuzzy $\widetilde{g}$-open set.

Definition 2.2. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,
(1) $\widetilde{g}$ $-I_T(\lambda, r) = \vee\{\mu \in I^X : \mu \leq \lambda,\ \mu \text{ is a r} - \text{fuzzy} \widetilde{\text{g}} - \text{openset}\}$ is called r-fuzzy $\widetilde{g}$-interior of

λ.
(2) $\widetilde{g}-C_T(\lambda,r)=\wedge\{\mu\in I^X:\mu\geq\lambda,\ \mu$ is a r $-$ fuzzy $\widetilde{\mathrm{g}}$ $-$closedset$\}$ is called r-fuzzy $\widetilde{g}$-closure of λ.

Definition 2.3. Let (X, T) be a smooth fuzzy topological space. For $\lambda\in I^X$ and $r\in I_0$, λ is called r-generalized fuzzy $\widetilde{g}$-closed (briefly, r-gf$\widetilde{g}$-closed) iff $\widetilde{g}-C_T(\lambda,r)\leq\mu$ whenever $\lambda\leq\mu,\ \mu\in I^X$ is r-fuzzy $\widetilde{g}$-open. The complement of a r-generalized fuzzy $\widetilde{g}$-closed set is a r-generalized fuzzy $\widetilde{g}$-open set (briefly, r-gf$\widetilde{g}$-open).

Definition 2.4. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f:(X,T)\to(Y,S)$ be a function.
(1) f is called $\widetilde{g}$-open (resp. $\widetilde{g}$-closed) if for each r-fuzzy $\widetilde{g}$-open set $\lambda\in I^X, f(\lambda)\in I^Y$ is r-fuzzy $\widetilde{g}$-open (resp. r-fuzzy $\widetilde{g}$-closed).
(2) f is called $\widetilde{g}$-continuous if for each $\lambda\in I^Y$ with $S(\lambda)\geq r, f^{-1}(\lambda)\in I^X$ is r-fuzzy $\widetilde{g}$-open.
(3) f is called fuzzy $\widetilde{g}$-irresolute if for each r-fuzzy $\widetilde{g}$-open set $\lambda\in I^Y, f^{-1}(\lambda)\in I^X$ is r-fuzzy $\widetilde{g}$-open.
(4) f is called fuzzy $\widetilde{g}$-homeomorphism if f is one to one, onto, fuzzy $\widetilde{g}$-irresolute and fuzzy $\widetilde{g}$-open.
(5) f is called gf$\widetilde{g}$-irresolute if for each r-gf$\widetilde{g}$ closed set $\lambda\in I^Y, f^{-1}(\lambda)\in I^X$ is r-gf$\widetilde{g}$-closed.
(6) f is called gf$\widetilde{g}$-closed iff for any r-gf$\widetilde{g}$-closed set $\lambda\in I^X$, $f(\lambda)$ is r- gf$\widetilde{g}$-closed.

Definition 2.5. A smooth fuzzy topological space (X, T) is called
(1) Fuzzy $\widetilde{g}-T_o$ iff for $\lambda,\ \mu\in I^X$ with $\lambda\ \overline{q}\ \mu$, there exists r-fuzzy $\widetilde{g}$-open set $\delta\in I^X$ such that either $\lambda\leq\delta$ or $\mu\leq\delta,\ \lambda\ \overline{q}\ \delta$.
(2) Fuzzy $\widetilde{g}-T_1$ iff for $\lambda,\ \mu\in I^X$ with $\lambda\ \overline{q}\ \mu$, there exist r-fuzzy $\widetilde{g}$-open sets $\delta,\ \eta\in I^X$ such that either $\lambda\leq\delta,\ \mu\ \overline{q}\ \delta$ or $\mu\leq\eta,\ \lambda\ \overline{q}\ \eta$.
(3) Fuzzy $\widetilde{g}-T_2$ iff for $\lambda,\ \mu\in I^X$ with $\lambda\ \overline{q}\ \mu$, there exist r-fuzzy $\widetilde{g}$-open sets $\delta,\ \eta\in I^X$ with $\lambda\leq\delta,\ \mu\leq\eta$ and $\delta\ \overline{q}\ \eta$.
(4) Fuzzy $\widetilde{g}-R_0$ iff $\lambda\ \overline{q}\ \widetilde{g}-C_T(\mu,r)$ implies that $\mu\ \overline{q}\ \widetilde{g}-C_T(\lambda,r)$ for $\lambda,\ \mu\in I^X$.

Definition 2.6. A smooth fuzzy topological space (X, T) is called fuzzy $\widetilde{g}-T_{1/2}$ if every r - gf$\widetilde{g}$-closed set is r-fuzzy $\widetilde{g}$- closed.

Proposition 2.1. Let (X, T) be a smooth fuzzy topological space. For r $\in I_0$, the following properties hold:
(i) For all r-fuzzy $\widetilde{g}$- open set $\lambda\in I^X$, λ q μ iff λ q ($\widetilde{g}-C_T\ (\mu,r)),\mu\in I^X$.
(ii) δ q $(\widetilde{g}-C_T(\lambda,r))$ iff λ q μ for all r-fuzzy $\widetilde{g}$ -open set $\mu\in I^X$ with $\delta\leq\mu$, for $\lambda,\ \delta\in I^X$.

Proof. (i) Let λ be a r-fuzzy $\widetilde{g}$-open set such that λ q μ. Since $\mu\leq\widetilde{g}-C_T(\mu,r)$, λ q $\widetilde{g}\,C_T(\mu,r)$. Conversely let λ be a r-fuzzy $\widetilde{g}$-open set such that $\lambda\ \overline{q}\ \mu$. Then $\mu\leq\overline{1}-\lambda$, this implies that $\widetilde{g}-C_T(\mu,r)\leq\widetilde{g}-C_T(\overline{1}-\lambda,r)=\overline{1}-\lambda$. Now,$\widetilde{g}-C_T(\mu,r)\leq\overline{1}-\lambda$. Thus $\lambda\ \overline{q}\ \widetilde{g}-C_T(\mu,r)$ which is a contradiction. Hence the result.

(ii) Let δ q $(\widetilde{g}-C_T(\lambda,r))$. Since $\delta\leq\mu$, μ q $(\widetilde{g}-C_T(\lambda,r))$. By (i), μ q λ for all r-fuzzy $\widetilde{g}$-open set μ with $\delta\leq\mu$. Conversely suppose that $\delta\ \overline{q}\ \widetilde{g}-C_T(\lambda,r)$. Then $\delta\leq\overline{1}-(\widetilde{g}-C_T(\lambda,r))$. Let $\mu=\overline{1}-(\widetilde{g}-C_T(\mu,r))$. Then μ is a r-fuzzy $\widetilde{g}$-open set. Since $\lambda\leq\widetilde{g}-C_T(\lambda,r),\mu=\overline{1}-(\widetilde{g}-C_T(\lambda,r))\leq\overline{1}-\lambda$, this implies that $\lambda\ \overline{q}\ \mu$, a contradiction. Hence the result.

Proposition 2.2. Let (X, T) be a smooth fuzzy topological space. For $\delta,\ \rho\in I^X$, the following statements are equivalent:

(i) (X, T) is fuzzy $\widetilde{g}-R_0$.
(ii) If $\delta\ \overline{q}\ \lambda = \widetilde{g} - C_T(\lambda, r), \lambda \in I^X$ and r $\in I_0$, there exists a r-fuzzy $\widetilde{g}$-open set μ such that $\delta\ \overline{q}\ \mu$ and $\lambda \leq \mu$.
(iii) If $\delta\ \overline{q}\ \lambda = \widetilde{g} - C_T(\lambda, r)$, then $\widetilde{g} - C_T(\delta, r)\ \overline{q}\ \lambda = \widetilde{g} - C_T(\lambda, r), \lambda \in I^X$ and $r \in I_0$.
(iv) If $\delta\ \overline{q}\ \widetilde{g} - C_T(\rho, r)$ then $\widetilde{g} - C_T(\delta, r)\ \overline{q}\ \widetilde{g} - C_T(\rho, r), r \in I_0$.

Proof. (i) $\Rightarrow$ (ii) Let $\delta\ \overline{q}\ \lambda = \widetilde{g} - C_T(\lambda, r)$. Since $\widetilde{g} - C_T(\rho, r) \leq \widetilde{g} - C_T(\lambda, r)$ for each $\rho \leq \lambda$, we have $\delta\ \overline{q}\ (\widetilde{g} - C_T(\rho, r))$. By (i), $\rho\ \overline{q}\ (\widetilde{g} - C_T(\delta, r))$. By (ii) of Proposition 2.1, for each $\rho\ \overline{q}\ (\widetilde{g} - C_T(\delta, r))$, there exists a r-fuzzy $\widetilde{g}$-open set η such that $\delta\ \overline{q}\ \eta$, $\rho \leq \eta$. Let $\mu = \vee\ \{\ \eta$: $\delta\ \overline{q}\ \eta\ \}$. Then $\delta\ \overline{q}\ \mu$, $\lambda \leq \mu$ for all r-fuzzy $\widetilde{g}$-open set μ.

(ii) $\Rightarrow$ (iii) Let $\delta\ \overline{q}\ \lambda = \widetilde{g} - C_T(\lambda, r)$. By (ii), there exists a r-fuzzy $\widetilde{g}$-open set $\mu \in I^X$ such that $\delta\ \overline{q}\ \mu$ and $\lambda \leq \mu$. Since $\delta\ \overline{q}\ \mu$, it follows that $\delta \leq \overline{1} - \mu$. This implies that $\widetilde{g} - C_T(\delta, r) \leq \widetilde{g} - C_T(\overline{1} - \mu, r) = \overline{1} - \mu \leq \overline{1} - \lambda$. Hence $\widetilde{g} - C_T(\delta, r)\ \overline{q}\ \lambda = \widetilde{g} - C_T(\lambda, r)$.

(iii)$\Rightarrow$(iv) Let $\delta\ \overline{q}\ \widetilde{g} - C_T(\rho, r)$. Since $\widetilde{g} - C_T(\widetilde{g} - C_T(\rho, r), r) = \widetilde{g} - C_T(\rho, r)$ and by (iii), $\widetilde{g} - C_T(\delta, r)\ \overline{q}\ \widetilde{g} - C_T(\rho, r)$.

(iv) $\Rightarrow$ (i) Let $\delta\ \overline{q}\ \widetilde{g} - C_T(\rho, r)$. By (iv), $\widetilde{g} - C_T(\delta, r)\ \overline{q}\ \widetilde{g} - C_T(\rho, r)$. Since $\rho \leq \widetilde{g} - C_T(\rho, r)$, $\rho\ \overline{q}\ \widetilde{g} - C_T(\delta, r)$. Hence (X, T) is fuzzy $\widetilde{g} - R_0$.

Proposition 2.3. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be a fuzzy $\widetilde{g}$-irresolute, gf$\widetilde{g}$-irresolute and fuzzy $\widetilde{g}$-closed function. Then the following conditions hold:
(i) If f is injective and (Y, S) is a fuzzy $\widetilde{g} - T_{1/2}$ space, then (X, T) is a fuzzy $\widetilde{g} - T_{1/2}$ space.
(ii) If f is surjective and (X, T) is a fuzzy $\widetilde{g} - T_{1/2}$ space, then (Y, S) is a fuzzy $\widetilde{g} - T_{1/2}$ space.

Proof. (i) Let $\lambda \in I^X$ be a r-gf$\widetilde{g}$-closed set. Since f is gf$\widetilde{g}$-closed, f (λ) $\in I^Y$ is r-gf$\widetilde{g}$-closed. Since (Y, S) is fuzzy $\widetilde{g} - T_{1/2}$, $f(\lambda)$ is r-fuzzy$\widetilde{g}$-closed. Now, $\lambda = f^{-1}(f(\lambda))$ is r-fuzzy $\widetilde{g}$-closed. Hence (X, T) is a fuzzy $\widetilde{g} - T_{1/2}$ space.

(ii) Let $\mu \in I^Y$ be a r-gf$\widetilde{g}$-closed set. Since f is gf$\widetilde{g}$-irresolute, $f^{-1}(\mu) \in I^X$ is a r-gf$\widetilde{g}$-closed set. Since (X, T) is a fuzzy $\widetilde{g} - T_{1/2}$ space, $f^{-1}(\mu)$ is a r-fuzzy $\widetilde{g}$-closed set. Therefore $\mu = f(f^{-1}(\mu))$ is r-fuzzy $\widetilde{g}$-closed. Hence (Y, S) is a fuzzy $\widetilde{g} - T_{1/2}$ space.

Proposition 2.4. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f : (X, T) \to (Y, S)$ be a fuzzy $\widetilde{g}$-irresolute, and injective function. If (Y, S) is fuzzy $\widetilde{g} - T_2$ (resp. fuzzy $\widetilde{g} - T_1$), then (X, T) is fuzzy $\widetilde{g} - T_2$ (resp. fuzzy $\widetilde{g} - T_1$).

Proof. Let (Y, S) be a fuzzy $\widetilde{g} - T_2$ space. Let $\lambda_1, \lambda_2 \in I^X$ be such that $\lambda_1\ \overline{q}\ \lambda_2$, then exist r-fuzzy $\widetilde{g}$-open sets λ, $\mu \in I^Y$ such that $f(\lambda_1) \leq \lambda$ and $f(\lambda_2) \leq \mu$ such that $\lambda\ \overline{q}\ \mu$. Then $\lambda \leq \overline{1} - \mu$ which implies that $f^{-1}(\lambda)\ \overline{q}\ f^{-1}(\mu)$. Now, $\lambda_1 \leq f^{-1}(\lambda)$ and $\lambda_2 \leq f^{-1}(\mu)$. Since f is fuzzy $\widetilde{g}$-irresolute, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are r-fuzzy $\widetilde{g}$-open sets. Hence (X, T) is a fuzzy $\widetilde{g} - T_2$ space. Similarly we prove the case of fuzzy $\widetilde{g} - T_1$ space.

§3. Fuzzy $\widetilde{g}$-normal spaces and its characterizations

In this section, the concept of fuzzy $\widetilde{g}$-normal space is introduced. Interesting properties and characterizations of such space are discussed.

Definition 3.1. A smooth fuzzy topological space (X, T) is said to be fuzzy $\widetilde{g}$-normal if for every r-fuzzy $\widetilde{g}$-closed set λ and r-fuzzy $\widetilde{g}$-open set μ with $\lambda \leq \mu$ there exists a $\gamma \in I^X$ such

that $\lambda \leq \widetilde{g} - I_T(\gamma, r) \leq \widetilde{g} - C_T(\gamma, r) \leq \mu, r \in I_0$.

Proposition 3.1. For any smooth fuzzy topological space (X, T) and $\lambda, \mu, \delta \in I^X$, $r \in I_0$, the following statements are equivalent:
(i) (X, T) is fuzzy $\widetilde{g}$-normal.
(ii) For each r-fuzzy $\widetilde{g}$-closed set λ and each r-fuzzy $\widetilde{g}$-open set μ with $\lambda \leq \mu$, there exists a r-fuzzy $\widetilde{g}$-open set δ such that $\widetilde{g} - C_T(\lambda, r) \leq \delta \leq \widetilde{g} - C_T(\delta, r) \leq \mu$.
(iii) For each r-gf$\widetilde{g}$-closed set λ and r-fuzzy $\widetilde{g}$-open set μ with $\lambda \leq \mu$, there exists a r-fuzzy $\widetilde{g}$-open set δ such that $\widetilde{g} - C_T(\lambda, r) \leq \delta \leq \widetilde{g} - C_T(\delta, r) \leq \mu$.

Proof. (i) $\Rightarrow$ (ii) The proof is trivial.

(ii) $\Rightarrow$ (iii) Let λ be any r-gf$\widetilde{g}$-closed set and μ be any r-fuzzy $\widetilde{g}$-open set such that $\lambda \leq \mu$. Since λ is r-gf$\widetilde{g}$-closed, $\widetilde{g} - C_T(\lambda, r) \leq \mu$. Now, $\widetilde{g} - C_T(\lambda, r)$ is r-fuzzy $\widetilde{g}$-closed and μ is r-fuzzy $\widetilde{g}$-open. By (ii), there exists a r-fuzzy $\widetilde{g}$-open set δ such that $\widetilde{g} - C_T(\lambda, r) \leq \delta \leq \widetilde{g} - C_T(\delta, r) \leq \mu$.

(iii) $\Rightarrow$ (i) The proof is trivial.

Proposition 3.2. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \to (Y, S)$ is fuzzy $\widetilde{g}$-homeomorphism and (Y, S) is fuzzy $\widetilde{g}$-normal, then (X, T) is fuzzy $\widetilde{g}$-normal.

Proof. Let $\lambda \in I^X$ be any r-fuzzy $\widetilde{g}$-closed set and $\mu \in I^X$ be any r-fuzzy $\widetilde{g}$-open set such that $\lambda \leq \mu$ where $r \in I_0$. Since f is fuzzy $\widetilde{g}$-homeomorphism, it is also fuzzy $\widetilde{g}$-closed. Hence $f(\lambda) \in I^Y$ is r-fuzzy $\widetilde{g}$-closed. Since f is fuzzy $\widetilde{g}$-open, $f(\mu) \in I^Y$ is r-fuzzy $\widetilde{g}$-open. Since (Y, S) is fuzzy $\widetilde{g}$-normal, there exists a $\gamma \in I^Y$ such that $f(\lambda) \leq \widetilde{g} - I_T(\gamma, r) \leq \widetilde{g} - C_T(\gamma, r) \leq f(\mu)$. Now, $f^{-1}(f(\lambda)) = \lambda \leq f^{-1}(\widetilde{g} - I_T(\gamma, r)) \leq f^{-1}(\widetilde{g} - C_T(\gamma, r)) \leq f^{-1}(f(\mu)) = \mu$. That is, $\lambda \leq \widetilde{g} - I_T(f^{-1}(\gamma), r) \leq \widetilde{g} - C_T(f^{-1}(\gamma), r) \leq \mu$. Therefore (X, T) is fuzzy $\widetilde{g}$-normal.

Proposition 3.3. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \to (Y, S)$ is fuzzy $\widetilde{g}$-homeomorphism and (X, T) is a fuzzy $\widetilde{g}$-normal space, then (Y, S) is fuzzy $\widetilde{g}$-normal.

Proof. Let $\lambda \in I^Y$ be any r-fuzzy $\widetilde{g}$-closed set and $\mu \in I^Y$ be any r-fuzzy $\widetilde{g}$-open set such that $\lambda \leq \mu$ where r $\in I_0$. Since f is fuzzy $\widetilde{g}$-irresolute, $f^{-1}(\lambda)$ is r-fuzzy $\widetilde{g}$-closed and $f^{-1}(\mu) \in I^Y$ is r-fuzzy $\widetilde{g}$-open. Since (X, T) is fuzzy $\widetilde{g}$-normal, there exists a $\gamma \in I^X$ such that $f^{-1}(\lambda) \leq \widetilde{g} - I_T(\gamma, r) \leq \widetilde{g} - C_T(\gamma, r) \leq f^{-1}(\mu)$. Now, $f(f^{-1}(\lambda)) = \lambda \leq f(\widetilde{g} - I_T(\gamma, r)) \leq f(\widetilde{g} - C_T(\gamma, r)) \leq f(f^{-1}(\mu)) = \mu$. That is, $\lambda \leq \widetilde{g} - I_T(f(\gamma), r) \leq \widetilde{g} - C_T(f(\gamma), r) \leq \mu$. Therefore (Y, S) is fuzzy $\widetilde{g}$-normal.

Proposition 3.4. Let (X, T) be a smooth fuzzy topological space which is also a fuzzy $\widetilde{g}$-normal space. Let $\{ \lambda_i \}_{i \in J} \in I^X$ and $\{ \mu_j \}_{j \in J} \in I^X$. If there exist $\lambda, \mu \in I^X$ such that $\widetilde{g} - C_T(\lambda_i, r) \leq \widetilde{g} - C_T(\lambda, r) \leq \widetilde{g} - I_T(\mu_j, r)$ and $\widetilde{g} - C_T(\lambda_i, r) \leq \widetilde{g} - I_T(\mu, r) \leq \widetilde{g} - I_T(\mu_j, r)$ for all $i, j = 1, 2, \cdots$, and r $\in I_0$, then there exists $\gamma \in I^X$ such that

$$\widetilde{g} - C_T(\lambda_i, r) \leq \widetilde{g} - I_T(\gamma, r) \leq \widetilde{g} - C_T(\gamma, r) \leq \widetilde{g} - I_T(\mu_j, r), \text{ for all } i, j = 1, 2, \cdots .$$

Proof. First, we shall show by induction that for all $n \geq 2$ there exists a collection $\{\gamma_i, \delta_i / 1 \leq i \leq n\}$ contained in I^X such that the conditions

$$\left.\begin{array}{l}\widetilde{g}-C_T(\lambda_i, r) \leq \widetilde{g}-I_T(\gamma_i, r);\\ \widetilde{g}-C_T(\delta_j, r) \leq \widetilde{g}-I_T(\mu_j, r);\\ \widetilde{g}-C_T(\lambda, r) \leq \widetilde{g}-I_T(\delta_j, r);\\ \widetilde{g}-C_T(\gamma_i, r) \leq \widetilde{g}-I_T(\mu, r);\\ \widetilde{g}-C_T(\gamma_i, r) \leq \widetilde{g}-I_T(\delta_j, r),\end{array}\right\} (s_n)$$

hold for all $i, j = 1, 2, ...n-1$. Clearly (S_2) follows at once from the fuzzy $\widetilde{g}$-normality of (X, T). Now, suppose that for $n \geq 2$, γ_i, $\delta_i \in I^X (i < n)$ such that (S_n) holds. Since $\widetilde{g}-C_T(\lambda_n, r) \leq \widetilde{g}-C_T(\lambda, r) \leq \widetilde{g}-I_T(\delta_j, r)(j < n)$ and $\widetilde{g}-C_T(\lambda_n, r) \leq \widetilde{g}-I_T(\mu, r)$ by fuzzy $\widetilde{g}$-normality of (X, T), there exists $\gamma_n \in I^X$ such that

$$\widetilde{g}-C_T(\lambda_n, r) \leq \widetilde{g}-I_T(\gamma_n, r) \leq \widetilde{g}-C_T(\gamma_n, r) \leq \widetilde{g}-I_T\left(\underset{j<n}{\wedge} \delta_j \wedge \mu,\ \mathrm{r}\right).$$

Similarly, since $\widetilde{g}-C_T(\lambda, r) \leq \widetilde{g}-I_T(\mu_n, r)$ and $\widetilde{g}-C_T(\gamma_i, r) \leq \widetilde{g}-I_T(\mu_n, r)(i \leq n)$, there exists $\delta_n \in I^X$ such that

$$\begin{aligned}\left(\underset{i\leq n}{\vee} \widetilde{g}-C_T(\gamma_i, r) \vee \widetilde{g}-C_T(\lambda, r)\right) &\leq \widetilde{g}-I_T(\delta_n, r)\\ &\leq \widetilde{g}-C_T(\delta_n, r)\\ &\leq \widetilde{g}-I_T(\mu_n, r).\end{aligned}$$

Thus (S_{n+1}) holds.

Let $\gamma = \overset{\infty}{\underset{i=1}{\vee}} \gamma_i$. Then $\widetilde{g}-C_T(\lambda_i, r) \leq \widetilde{g}-I_T(\gamma_i, r) \leq \widetilde{g}-I_T(\gamma, r)$ for all $i = 1, 2, \cdots$. Since $\widetilde{g}-C_T(\gamma_i, r) \leq \widetilde{g}-I_T(\delta_j, r)(i, j = 1, 2, \cdots), \gamma_i \leq \delta_j$, so that $\widetilde{g}-C_T(\gamma, r) \leq \widetilde{g}-C_T(\delta_j, r) \leq \widetilde{g}-I_T(\mu_j, r)$ for all $j = 1, 2, \cdots$. This proves the result.

Proposition 3.5. Let (X, T) be a smooth fuzzy topological space which is also a fuzzy $\widetilde{g}$-normal space. If $\{\lambda_q\}_{q\in Q}$ and $\{\mu_q\}_{q\in Q}$ are monotone increasing collections of respectively, fuzzy $\widetilde{g}$-closed and fuzzy $\widetilde{g}$-open subsets of (X, T) (Q is the set of all rational numbers) such that $\lambda_q \leq \mu_s$ whenever $q < s$, then there exists a collection $\{\gamma_q\}_{q\in Q} \in I^X$ such that

$$\lambda_q \leq \widetilde{g}-I_T(\gamma_s, r), \widetilde{g}-C_T(\gamma_q, r) \leq \widetilde{g}-I_T(\gamma_s, r) \text{ and } \widetilde{g}-C_T(\gamma_q, r) \leq \mu_s$$

whenever $q < s$.

Proof. Let us arrange into a sequence q_n of all rational numbers (without repetitions). For every $n \geq 2$ we shall define inductively a collection $\{\gamma_{q_i}/1 \leq i \leq n\} \in I^X$ such that

$$\left.\begin{array}{ll}\lambda_q \leq \widetilde{g}-I_T(\gamma_{q_i}, r), & \text{if } q < q_i;\\ \widetilde{g}-C_T(\gamma_{q_i}, r) \leq \mu_q, & \text{if } q_i < q;\\ \widetilde{g}-C_T(\gamma_{q_i}, r) \leq \widetilde{g}-I_T(\gamma_{q_j}, r), & \text{if } q_i < q_j,\end{array}\right\} (s_n)$$

for all $1 \leq i, j < n$. It is clear that the countable collections $\{\lambda_q/q < q_1\}$ and $\{\mu_q/q > q_1\}$ together with λ_{q_1} and μ_{q_1} satisfy all hypotheses of Proposition 3.4, so that there exists $\delta_1 \in I^X$ such that $\lambda_q \leq \widetilde{g}-I_T(\delta_1, r)$ for all $q < q_1$ and $\widetilde{g}-C_T(\delta_1, r) \leq \mu_q$ for all $q > q_1$.

Letting $\gamma_{q_1} = \delta_1$, we get (S_2). Assume that the fuzzy subsets γ_{q_i} are already defined for $i < n$ and satisfy (S_n). Define

$$\lambda = \vee\{\gamma_{q_i}/i < n, q_i < q_n\} \vee \lambda_{q_n} \text{ and } \mu = \wedge\{\gamma_{q_j}/j < n, q_j < q_n\} \wedge \mu q_n.$$

Then $\widetilde{g}-C_T(\gamma_{q_i}, r) \leq \widetilde{g}-C_T(\lambda, r) \leq \widetilde{g}-I_T(\gamma_{q_j}, r)$, and $\widetilde{g}-C_T(\gamma_{q_i}, r) \leq \widetilde{g}-I_T(\mu, r) \leq \widetilde{g}-I_T(\gamma_{q_j}, r)$ whenever $q_i < q_n < q_j, (i, j < n)$ as well as $\lambda_q \leq \widetilde{g}-C_T(\lambda, r) \leq \mu_s$ and $\lambda_q \leq \widetilde{g}-I_T(\mu, r) \leq \mu_s$ whenever $q < q_n < s$. This shows that the countable collections $\{\gamma_{q_i}/i <, q_i < q_n\} \vee \{\lambda_q/q < q_n\}$ and $\{\gamma_{q_j}/j < n, q_j > q_n\} \vee \{\mu_q/q > q_n\}$ together with λ and μ satisfy all hypotheses of Proposition 3.5. Hence there exists a $\delta_n \in I^X$ such that

$$\begin{array}{ll} \lambda_q \leq \widetilde{g}-I_T(\delta_n, r), & \text{if } q < q_n \\ \widetilde{g}-C_T(\gamma_{q_i}, r) \leq \widetilde{g}-I_T(\delta_n, r), & \text{if } q_i < q_n \\ \widetilde{g}-C_T(\delta_n, r) \leq \mu_q, & \text{if } q_n < q \\ \widetilde{g}-C_T(\delta_n, r) \leq \widetilde{g}-I_T(\gamma_{q_j}, r), & \text{if } q_n < q_j \end{array}$$

where $1 \leq i, j \leq n-1$. Letting $\gamma_{q_n} = \delta_n$ we obtain fuzzy subsets $\gamma_{q_1}, \gamma_{q_2}, \cdots, \gamma_{q_n}$ that satisfy the result (S_{n+1}). Therefore the collection $\{\gamma_{q_i}/i = 1, 2, \cdots.\}$ has the required properties. This completes the proof.

§4. Fuzzy $\widetilde{g}$-regular space and its characterizations

In this section, the concept of fuzzy $\widetilde{g}$-regular space is introduced. Some interesting characterizations are established.

Definition 4.1. A smooth fuzzy topological space (X, T) is called a fuzzy $\widetilde{g}$-regular space if for every r-fuzzy $\widetilde{g}$-closed set λ and each $\alpha \in I^X$ with $\alpha \not\leq \lambda$, there exist $\mu, \delta \in I^X$ with $T(\mu) \geq r, T(\delta) \geq r$ and $\delta\ \overline{q}\ \mu$ such that $\alpha \leq \delta$, $\lambda \leq \mu$.

Proposition 4.1. Let (X, T) be a smooth fuzzy topological space. Then the following statements are equivalent:
(i) (X, T) is fuzzy $\widetilde{g}$-regular.
(ii) For each $\alpha \in I^X$ and r-fuzzy $\widetilde{g}$-open set λ with α q λ there exists a $\delta \in I^X$ with $T(\delta) \geq r, \alpha \leq \delta$ such that $C_T(\delta, r) \leq \lambda$.

Proof. (i) $\Rightarrow$ (ii) Let λ be any r-fuzzy $\widetilde{g}$-open set with α q λ. By hypothesis, there exist $\mu, \delta \in I^X$ with $T(\mu) \geq r, T(\delta) \geq r$ and $\delta\ \overline{q}\ \mu$ such that $\overline{1} - \lambda \leq \mu$ and $\alpha \leq \delta$. Since $\delta \leq \overline{1} - \mu, C_T(\delta, r) \leq C_T(\overline{1} - \mu, r) = \overline{1} - \mu$. But $\overline{1} - \lambda \leq \mu$ gives $\overline{1} - \mu \leq \lambda$. That is, $C_T(\delta, r) \leq \lambda$. Hence the result.

(ii) $\Rightarrow$ (i) Let γ be any r-fuzzy $\widetilde{g}$-closed set with $\alpha \not\leq \gamma$ for any $\alpha \in I^X$. Now, $\overline{1} - \gamma$ is r-fuzzy $\widetilde{g}$-open. By hypothesis, there exists a $\delta \in I^X$ with $T(\delta) \geq r, \alpha \leq \delta$ such that $C_T(\delta, r) \leq \overline{1} - \gamma$. Then $\gamma \leq \overline{1} - C_T(\delta, r)$. Now, $\delta \leq \overline{1} - (\overline{1} - C_T(\delta, r))$ such that $\alpha \leq \delta$ and $\gamma \leq \overline{1} - C_T(\delta, r)$. Therefore (X, T) is fuzzy $\widetilde{g}$-regular.

Proposition 4.2. Let (X, T) be a smooth fuzzy topological space. Then (X, T) is fuzzy $\widetilde{g}$-regular iff for every r-fuzzy $\widetilde{g}$-closed set $\lambda \in I^X$ and $\alpha \in I^X$ with $\alpha \not\leq \lambda$, there exist $\mu, \delta \in I^X$ with $T(\mu) \geq r$, $T(\delta) \geq r$ such that $\alpha \leq \delta$, $\lambda \leq \mu$, then $\mu\ \overline{q}\ C_T(\delta, r)$ where $r \in I_0$.

Proof. Let (X, T) be a fuzzy $\widetilde{g}$-regular space. Let λ be any r-fuzzy $\widetilde{g}$-closed set and α be such that $\alpha \nleq \lambda$. Since (X, T) is fuzzy $\widetilde{g}$-regular, there exist μ, δ with $T(\mu) \geq r, T(\delta) \geq r, \delta \; \overline{q} \; \mu$ such that $\alpha \leq \delta$, $\lambda \leq \mu$. Now, $\delta \; \overline{q} \; \mu$ implies that $C_T(\delta, r) \leq C_T(\overline{1} - \mu, r) = \overline{1} - \mu$. That is, $\mu \; \overline{q} \; C_T(\delta, r)$. Hence the result. Converse part is trivial.

Proposition 4.3. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \to (Y, S)$ is bijective, fuzzy $\widetilde{g}$-irresolute, fuzzy open and if (X, T) is a fuzzy $\widetilde{g}$-regular space, then (Y, S) is fuzzy $\widetilde{g}$-regular.

Proof. Let $\lambda \in I^Y$ be any r-fuzzy $\widetilde{g}$-closed set and $\beta \in I^Y$ be such that $\beta \nleq \lambda$, $r \in I_0$. Since f is fuzzy $\widetilde{g}$-irresolute, $f^{-1}(\lambda) \in I^X$ is r-fuzzy $\widetilde{g}$-closed. Let $f(\alpha) = \beta$ for any $\alpha \in I^X$. Since f is bijective, $\alpha = f^{-1}(\beta)$. Since (X, T) is fuzzy $\widetilde{g}$-regular and $\alpha \nleq f^{-1}(\lambda)$ there exist $\mu\delta \in I^X$ with $T(\mu) \geq r, T(\delta) \geq r$ and $\delta \; \overline{q} \; \mu$ such that $\alpha \leq \delta$ and $f^{-1}(\lambda) \leq \mu$. Since f is fuzzy open and bijective, $f(\alpha) \leq f(\delta)$ implies that $\beta \leq f(\delta), \lambda \leq f(\mu)$ and $S(f(\delta)) \geq r, S(f(\mu)) \geq r$ with $f(\delta) \; \overline{q} \; f(\mu)$. Hence (Y, S) is fuzzy $\widetilde{g}$-regular.

Proposition 4.4. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \to (Y, S)$ is fuzzy $\widetilde{g}$-closed, fuzzy continuous, injective and (Y, S) is fuzzy $\widetilde{g}$-regular then (X, T) is fuzzy $\widetilde{g}$-regular.

Proof. Let $\lambda \in I^X$ be any r-fuzzy $\widetilde{g}$-closed set and $\alpha \in I^X$ be such that $\alpha \nleq \lambda$, $r \in I_0$. Since f is fuzzy $\widetilde{g}$-closed, $f(\lambda) \in I^Y$ is r-fuzzy $\widetilde{g}$-closed and $f(\alpha) \nleq f(\lambda)$. Since (Y, S) is fuzzy $\widetilde{g}$-regular, there exist $\mu, \delta \in I^Y$ with $S(\mu) \geq r$, $S(\delta) \geq r$ and $\delta \; \overline{q} \; \mu$ such that $f(\alpha) \leq \mu$ and $f(\lambda) \leq \delta$. Since f is fuzzy continuous, $f^{-1}(\mu), f^{-1}(\delta) \in I^X$ with $T(f^{-1}(\mu)) \geq r$ and $T(f^{-1}(\delta)) \geq r$. Also, $\alpha \leq f^{-1}(\mu), \lambda \leq f^{-1}(\delta)$ and $f^{-1}(\delta) \; \overline{q} \; f^{-1}(\mu)$. Therefore (X, T) is fuzzy $\widetilde{g}$-regular.

Proposition 4.5. Let (X, T) be a smooth fuzzy topological space. Then the following statements are equivalent:

(i) (X, T) is fuzzy $\widetilde{g}$-regular.

(ii) For every r-fuzzy $\widetilde{g}$-open set λ such that $\alpha \leq \lambda$ there exists a $\gamma \in I^X$ with $T(\gamma) \geq r$ such that $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda$.

(iii) For every r-fuzzy $\widetilde{g}$-open set λ such that $\alpha \leq \lambda$ there exists a $\delta \in I^X$ with $T(\delta) \geq r$ and $\delta = I_T(\Delta, r), T(\overline{1} - \Delta) \geq r$ such that $\alpha \leq \delta \leq C_T(\delta, r) \leq \lambda$.

(iv) For every r-fuzzy $\widetilde{g}$-closed set μ such that $\alpha \nleq \mu$ there exist γ and λ with $T(\gamma) \geq r$ and $T(\lambda) \geq r$ such that $\alpha \leq \gamma$, $\mu \leq \lambda$ with $C_T(\gamma, r) \; \overline{q} \; C_T(\lambda, r)$.

Proof. (i) $\Rightarrow$ (ii) Let λ be a r-fuzzy$\widetilde{g}$-open set such that $\alpha \leq \lambda$. Then $\overline{1} - \lambda$ is a r-fuzzy $\widetilde{g}$-closed set such that $\alpha \nleq \overline{1} - \lambda$. Since (X, T) is $\widetilde{g}$-regular, there exist γ, $\delta \in I^X$ with $T(\gamma) \geq r$, $T(\delta) \geq r$ and $\gamma \; \overline{q} \; \delta$ such that $\alpha \leq \gamma$, $\overline{1} - \lambda \leq \delta$. Since $\gamma \; \overline{q} \; \delta$, $\gamma \leq \overline{1} - \delta$. Hence $C_T(\gamma, r) \leq C_T(\overline{1} - \delta, r) = \overline{1} - \delta$. But $\overline{1} - \delta \leq \lambda$. Therefore $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda$.

(ii) $\Rightarrow$ (iii) Let λ be a r-fuzzy $\widetilde{g}$-open set such that $\alpha \leq \lambda$. By (ii), there exists a $\gamma \in I^X$ with $T(\gamma) \geq r$ such that $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda$. Let $\delta = I_T(\Delta, r)$ where $\Delta = C_T(\gamma, r)$. Now, $\alpha \leq \gamma \leq I_T(\Delta, r) \leq C_T(\Delta, r) \leq \lambda$. Also, then $\alpha \leq \delta \leq C_T(\delta, r) = C_T(I_T(\Delta, r), r) \leq C_T(\Delta, r) = C_T(C_T(\gamma, r), r) = C_T(\gamma, r) \leq \lambda$. Thus $\alpha \leq \delta \leq C_T(\delta, r) \leq \lambda$.

(iii) $\Rightarrow$ (iv) Let μ be a r-fuzzy $\widetilde{g}$-closed set with $\alpha \nleq \mu$. Then $\overline{1}-\mu$ is a r-fuzzy $\widetilde{g}$-open set with $\alpha \leq \overline{1} - \mu$. By (iii), there exists a $\delta \in I^X$ with $T(\delta) \geq r$ such that $\alpha \leq \delta \leq C_T(\delta, r) \leq \overline{1} - \mu$ where $\delta = I_T(\Delta, r)$ for some $\Delta \in I^X$ with $T(\overline{1} - \Delta) \geq r$. Again by hypothesis there exists a γ

$\in I^X$ such that $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \delta$. Let $\lambda = \overline{1} - C_T(\delta, r)$. Then $\alpha \leq \gamma$, $\mu \leq \lambda$ with $\lambda \leq \overline{1} - \delta$. Now, $C_T(\lambda, r) \leq \overline{1} - \delta \leq \overline{1} - C_T(\gamma, r)$. Thus $C_T(\gamma, r)\ \overline{q}\ C_T(\lambda, r)$.

(iv) $\Rightarrow$ (i) The proof is trivial.

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An integral identity involving the Hermite polynomials[1]

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Abstract The main purpose of this paper is using the elementary method and the properties of the power series to study an integral calculating problem involving the Hermite polynomials, then give an interesting identity.

Keywords Hermite polynomials, integral calculation, identity, elementary method.

§1. Introduction

For any real number x, the polynomial solutions

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \sum_{0 \le k \le \frac{n}{2}} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}$$

of the Hermite equations

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \quad (n = 0,\ 1,\ 2,\ \cdots)$$

are called Hermite polynomials, see [1]. For example, the first several polynomials are: $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$, $H_5(x) = 32x^5 - 160x^3 + 120x$, $\cdots$. It is well know that $H_n(x)$ is an orthogonality polynomial. That is,

$$\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0, & \text{if } m \neq n; \\ 2^n n! \sqrt{\pi}, & \text{if } m = n. \end{cases}$$

And it play a very important rule in the theories and applications of mathematics. So there are many people had studied its properties, some results and related papers see references [2], [3], [4], [5] and [6].

In this paper, we shall study the calculating problem of the integral

$$\sum_{a_1+a_2+\cdots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} H_{a_1}(x) H_{a_2}(x) \cdots H_{a_k}(x)\ dx, \tag{1}$$

and give an interesting calculating formula for it. About this problem, it seems that no one had studied yet, at least we have not seen any related papers before. The problem is interesting,

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because it can help us to know more information about the orthogonality of $H_n(x)$. The main purpose of this paper is using the elementary method and the properties of the power series to give an exact calculating formula for (1). That is, we shall prove the following conclusions:

Theorem 1. Let n and k are two positive integer with $n \geq k$, then we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \cdots \frac{H_{a_k}(x)}{a_k!}\, dx = \begin{cases} 0, & \text{if } n = 2m-1; \\ \frac{\sqrt{\pi}(k^2-k)^m}{m!}, & \text{if } n = 2m. \end{cases}$$

where $\sum\limits_{a_1+a_2+\cdots+a_k=n}$ denotes the summation over all nonnegative integers $a_1,\ a_2,\ \cdots,\ a_k$ such that $a_1 + a_2 + \cdots a_k = n$.

Theorem 2. Let m, n and k are positive integers with $n \geq k \geq 1$, then we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H^{(m)}_{a_1+m}(x)}{(a_1+m)!} \frac{H^{(m)}_{a_2+m}(x)}{(a_2+m)!} \cdots \frac{H^{(m)}_{a_k+m}(x)}{(a_k+m)!}\, dx$$

$$= \begin{cases} 0, & \text{if } n = 2m-1; \\ \frac{\sqrt{\pi}\cdot 2^{km}\cdot(k^2-k)^m}{m!}, & \text{if } n = 2m. \end{cases}$$

where $H_n^{(m)}(x)$ denotes the m-th derivative of $H_n(x)$ for x.

From Theorem 1 and Theorem 2 we know that the integration must be 0, if n be an odd number. So it is interesting that the orthogonality in such an integral only depend on the parity of n.

§2. Proof of the theorems

In this section, we shall use the elementary method and the properties of the power series to prove our Theorems directly. First we prove Theorem 1. For any positive integer k, from the generating function of $H_n(x)$ and the properties of the power series we have

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

and

$$e^{k2xt-kt^2} = \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_k=n} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \cdots \frac{H_{a_k}(x)}{a_k!} \right) t^n. \tag{2}$$

So from (2) we may get

$$\int_{-\infty}^{+\infty} e^{2kxt-kt^2-x^2} dx = \int_{-\infty}^{+\infty} e^{-(x-kt)^2+(k^2-k)t^2} dx$$

$$= \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\cdots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \cdots \frac{H_{a_k}(x)}{a_k!} dx \right) t^n. \tag{3}$$

On the other hand, for any real number t and integer x, note that the integral

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-(x-kt)^2+(k^2-k)t^2}dx &= e^{(k^2-k)t^2}\int_{-\infty}^{+\infty} e^{-(x-kt)^2}dx \\ &= e^{(k^2-k)t^2}\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}e^{(k^2-k)t^2} \end{aligned}\tag{4}$$

and

$$e^{(k^2-k)t^2} = \sum_{n=0}^{\infty}\frac{(k^2-k)^n}{n!}t^{2n}. \tag{5}$$

Combining (2), (3), (4) and (5) we may get

$$\begin{aligned}&\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(k^2-k)^n}{n!}t^{2n} \\ =& \sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}\int_{-\infty}^{+\infty} e^{-x^2}\frac{H_{a_1}(x)}{a_1!}\frac{H_{a_2}(x)}{a_2!}\cdots\frac{H_{a_k}(x)}{a_k!}dx\right)t^n.\end{aligned}\tag{6}$$

Comparing the coefficients of t^n in (6) we may immediately deduce the identity

$$\sum_{a_1+a_2+\cdots+a_k=n}\int_{-\infty}^{+\infty} e^{-x^2}\frac{H_{a_1}(x)}{a_1!}\frac{H_{a_2}(x)}{a_2!}\cdots\frac{H_{a_k}(x)}{a_k!}\,dx = \begin{cases} 0, & \text{if } n=2m-1; \\ \frac{\sqrt{\pi}(k^2-k)^m}{m!}, & \text{if } n=2m. \end{cases}$$

where $\sum\limits_{a_1+a_2+\cdots+a_k=n}$ denotes the summation over all nonnegative integers $a_1,\ a_2,\ \cdots,\ a_k$ such that $a_1+a_2+\cdots a_k=n$. This proves Theorem 1.

Now we prove Theorem 2. Note that

$$\frac{d^m}{dx^m}e^{2xt-t^2} = (2t)^m e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{H_{n+m}^{(m)}(x)}{(n+m)!}t^{n+m}$$

or

$$2^m e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{H_{n+m}^{(m)}(x)}{(n+m)!}t^n.$$

Then using the method of proving Theorem 1 we may get

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_k=n}\int_{-\infty}^{+\infty} e^{-x^2}\frac{H_{a_1+m}^{(m)}(x)}{(a_1+m)!}\frac{H_{a_2+m}^{(m)}(x)}{(a_2+m)!}\cdots\frac{H_{a_k+m}^{(m)}(x)}{(a_k+m)!}\,dx \\ &= \begin{cases} 0, & \text{if } n=2m-1; \\ \frac{\sqrt{\pi}\cdot 2^{km}\cdot(k^2-k)^m}{m!}, & \text{if } n=2m. \end{cases}\end{aligned}$$

where $H_n^{(m)}(x)$ denotes the m-th derivative of $H_n(x)$ for x.

This completes the proof of Theorem 2.

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Vinegar identification by ultraviolet spectrum technology and pattern recognition method

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Abstract Some kinds of vinegars are studied for identification. First, their ultraviolet spectrum curves are obtained by evaporation and ultraviolet spectrum scanning under the conditions of wavelength at 245∼ 330nm, quantity dilution ratio of the liquid at1:6, evaporation temperature at 45°C, mass concentration of reference at 45g/L. Then, the data are analyzed by the method of pattern recognition, such as Euclid (Mahalanobis) distance, linear discriminant analysis, principal component analysis, hybrid discriminant analysis and BP neural network. The identification accuracy of Euclid(Mahalanobis) distance, principal component analysis, hybrid discriminant analysis ($\lambda = 0, \eta = 1$) and BP are 100%. The results show that these methods can be effective ways to identify vinegar.

Keywords Uitraviolet Spectrum, Vinegar, identification, Hybrid Discriminant Analysis, BP Neural Network.

§1. Introduction

There are many varieties of vinegar in the market nowadays, but their qualities are not same, and there are not quick and valid methods to identify them. For a long period, people identify them from some sense index such as color, smelling, taste, style and some simple quantity index by experience; these methods are of subjectivity and unilateralism by all appearances.

Zhang Shunping et. al. measured vinegar by electronic nose technology, analyzing the comparability and similitude degree of vinegar at the aspect of savour, class and acidity by clustering and principal component analysis method, he also recognised the vinegar by probability neural network, the accuracy of recognition is 94.4% [1].

Ultraviolet Spectrum technology is a new measure technique, since there are different components in different matter systems and the unsaturation degree of the components are not same, the UV abs curves of the matter system are different. We can identify the matter systems by comparing their UV abs curves. In our previous works [2], we took the similitude degree of the curve as index, test the recurring, stability and otherness of the ultraviolet spectrum method,

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we concluded that this method can identify the vinegars. In this paper, we further study the method of vinegar identification, we process and analyzed the data of UV abs curves by pattern recognition method, such as Euclid(Mahalanobis) distance, linear discriminant analysis, principal component analysis, hybrid discriminant analysis and BP neural network. We recognize 5 kinds of vinegar samples, the accuracy of recognition is 100% when using Euclid(Mahalanobis) distance, principal component analysis, hybrid discriminant analysis ($\lambda = 0, \eta = 1$) and BP.

§2. The material and data

2.1. Experiment material

Vinegars: (a)Black Rice spicy vinegar, (b) Jiajia mature vinegar, (c)Shuita mature vinegar, (d)Xiaoerhei grain spicy vinegar, (e)Zhenjiang spicy vinegar, are all bought from Yangling Guomao Supermarket.

Self-made vinegar: the samples are from the Practice Factory of College of Food Science and Engineering, Northwest A and F University.

Reagent: Glacial acetic acid，sodium hydroxide, are all analysis pure reagent and made in China.

Water for experiment: distilled water.

2.2. Instrument for experiment:

BUCHI Rotavapor R-200 circumrotate evaporation instrument(BUCHI Company), UV-2550 double beam of ultraviolet—visible light photometer(Japan).

2.3. Condition of experiment and data

With the scanning wavelength range at 245∼330nm, samplinging interval at 0.5nm, aperture width at 0.5nm, dilution ratio of evaporated liquor at 1:6,evaporation temperature at 45°C，quality thickness of reference fluid (glacial acetic acid) at 45g/L,we scan the vinegars with ultraviolet spectrum at different storage time, and obtain the data, see [2].

We can see that, the UV abs curves of vinegar which have same brand are very similar, while the UV abs curves of vinegar which have different brand are of great difference. So we consider processing and analyzing the data by using the method of pattern recognition, and identify the vinegar.

With samplinging interval at 0.5nm, a vinegar sample is a vector with 171 dimension, while there are only 7 samples for one kind of vinegar, the dimension of sample vectors is far more than the number of samples, it is probably to appear severe warp when using statistical methods, so we take the interval as 5nm, and the dimension of sample vectors is now 18.

Table 1: The Identification Accuracy of Euclid(Mahalanobis) Distance

Name of vinegar	Zhenjiang spicy vinegar	Xiaoerhei grain spicy vinegar	Shuita mature vinegar	Jiajia mature vinegar	Black Rice spicy vinegar
Euclid distance	100%	100%	100%	100%	100%
Mahalanobis distance	100%	100%	100%	100%	100%

§3. process and analyze data by pattern recognition method

we randomly choose 5 samples from each kind of vinegar samples for training, the remained 2 samples for testing.

3.1. Euclid(Mahalanobis) distance method

This method is to classify the original samples. Let the ith training sample of the kth kind of vinegar be $X_i^k = (x_{i1}^k, x_{i2}^k, \ldots, x_{im}^k)^T, k = 1, 2, \ldots, 5, i = 1, 2, \ldots, 5, n = 18$, we calculate the mean vector (centre) of each training sample class:

$$m_k = \frac{1}{5}\sum_{i=1}^{5} x_i^k \tag{1}$$

and the covariance matrix:

$$s_k = \frac{1}{5}\sum_{i=1}^{5}(x_i^k - m_k)(x_i^k - m_k)^T \tag{2}$$

For each testing samples y, we calculate the Euclid distance between y and each centre:

$$d_k^E = [(y - m_k)^T (y - m_k)]^{\frac{1}{2}} \tag{3}$$

and the Mahalanobis distance:

$$d_k^M = [(y - m_k)^T s_k^{-1} (y - m_k)]^{\frac{1}{2}} \tag{4}$$

At last, assign the testing sample y to the 'nearest' class p as the following equation:

$$p = \underset{k}{argmin}\, d_k^E \quad or \quad p = \underset{k}{argmin}\, d_k^M \tag{5}$$

We test 10 testing samples by MATLAB7.4, the result is as Table 1:

3.2. Hybrid Discriminant Analysis(HDA) [3]

3.2.1. Method

HDA is to project the original samples data x (including training and testing samples) into a one-dimension subspace by the following linear transformation:

$$y = \mathbf{w}^T \mathbf{x} \tag{6}$$

Then the classifying is done in the subspace, the testing speed is quicker, and the accuracy of classification is still 100%. HDA is a method based on LDA and PCA. It integrates both discriminant and descriptive information simultaneously, controls the balance between LDA and PCA, it also provides a 2-D parameter space for searching. The objective function of HDA is:

$$\mathbf{w}_{opt} = \begin{matrix} argmax \\ \omega \end{matrix} \frac{|\mathbf{w}^T[(1-\lambda)S_b + \lambda S_\Sigma]\mathbf{w}|}{|\mathbf{w}^T[(1-\eta)S_\omega + \eta I]\mathbf{w}|} \tag{7}$$

where λ, η are tow parameters ranged from 0 to 1,

$$S_b = \sum_{k=1}^{5} 5(m_k - m)(m_k - m)^T \tag{8}$$

is the between-class scatter matrix, and $\mathbf{m} = \frac{1}{25}\Sigma\mathbf{x} = \frac{1}{25}\sum_{k=1}^{5} 5m_k$ is the mean vector of all training samples, while

$$S_\omega = \sum_{k=1}^{5} S_k, \quad and \quad S_k = \sum_{i=1}^{5} (x_i^k - m_k)(x_i^k - m_k)^T \tag{9}$$

is the within-class scatter matrix.

S_Σ is the covariance matrix of all training samples, $S_\Sigma = \frac{1}{25}\Sigma(\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^T$,$\mathbf{I}$ is unit matrix. According to the Lagrange function method, the solution $\mathbf{w}$ is the largest eigenvector (corresponding the largest eigenvalue) of $[(1-\eta)S_\omega + \eta I]^{-1}[(1-\lambda)S_b + \lambda S_\Sigma]$.

We project all the training and testing samples as (6), let the projection of training sample x_i^k be y_i^k , then the projection centre of each class in subspace is $m_k = \frac{1}{5}\sum_{i=1}^{5} y_i^k$, let y be the projection of a testing sample, we calculate the distance between y and the projection centre of each class $d_k = |y - m_k|$, assign the testing sample into the 'nearest'class.

3.2.2. Result and Discussion

Let λ, η be the different parameters between 0 and 1, by using MATLAB7.4, we test 10 testing samples, the result is as Table 2:

We have seen that when $\lambda = 1, \eta = 1$ or $\lambda = 0, \eta = 1$,the result is very satisfying, but it is poor when $\lambda = 0, \eta = 0$.

In fact, when $\lambda = 0, \eta = 0$, (7) becomes:

$$\mathbf{w}_{opt} = \begin{matrix} argmax \\ \omega \end{matrix} \frac{|\mathbf{w}^T S_b \mathbf{w}|}{|\mathbf{w}^T S_\omega \mathbf{w}|} \tag{10}$$

Table 2: The Identification Accuracy of HAD with Different Parameters

Name of vinegar	Zhenjiang spicy vinegar	Xiaoerhei grain spicy vinegar	Shuita mature vinegar	Jiajia mature vinegar	Black Rice spicy vinegar
$\lambda = 1, \eta = 1$	100%	100%	100%	100%	100%
$\lambda = 0, \eta = 1$	100%	100%	100%	100%	100%
$\lambda = 0, \eta = 0$	100%	100%	50%	100%	50%

This is the objective function of LDA, the solution $\mathbf{w}$ is now the largest eigenvector of $S_\omega^{-1} S_b$. Since the estimate of the scatter matrix S_ω and S_b is based on samples, LDA may not do well in small sample set problem. This conclusion is proved by the result of Table 2.
While $\lambda = 0, \eta = 1$, (7) becomes as following:

$$\mathbf{w}_{opt} = \begin{matrix} argmax \\ \omega \end{matrix} \frac{|\mathbf{w}^T S_b \mathbf{w}|}{|\mathbf{w}^T \mathbf{I} \mathbf{w}|} \tag{11}$$

The solution $\mathbf{w}$ is now the largest eigenvector of between-class scatter matrix S_b , this shows that (6) makes the scatter degree among each projection class largest, so the distance among each projection class is largest, the result of classification is good of course.

Let $\lambda = 1, \eta = 1$, then (7) becomes as following:

$$\mathbf{w}_{opt} = \begin{matrix} argmax \\ \omega \end{matrix} \frac{|\mathbf{w}^T S_{\sum} \mathbf{w}|}{|\mathbf{w}^T \mathbf{I} \mathbf{w}|} \tag{12}$$

is the objective function of PCA , $\mathbf{w}$, which is the solution of the objective function, is the eigenvector associated with the largest eigenvalue of the covariance matrix $S_{\sum}$. PAC is statistical analysis method, it can remove the relativity between the elements of the vector, so the components of the transformed vector are disrelated, and be arranged in the order that the corresponding variance are decreased. PAC is superior to LDA in dealing with the small sample set problems because it captures the descriptive information of the data in the projected space.This can be shown in Table 2.

For further analysis, we calculate the nonzero eigenvalues and the accumulated variance cover rate of the covariance matrix.

From the Table 3, we can see that the accumulated variance cover rate of the largest eigenvalue has arrived 93.22%. This shows that the variance contribution rate of the first principal component of the transformed vector has arrived 93.22%, so we obtain the satisfying results by only using the first principal component for classification.

The eigenvector associated with the largest eigenvalue is $\mathbf{w}$,
$\mathbf{w}$ =(-0.2976 -0.3487 -0.3781 -0.3691 -0.3676 -0.3157 -0.3499 -0.3105 -0.2069 -0.1092 -0.0457

Table 3: The Identification Accuracy of BP Neural Network

	1	2	3	4	5	6	7	8	9
Eigenvalues	20.6706	1.4026	0.046	0.0245	0.0141	0.0125	0.0043	0.0001	0.0001
Accumulated Cover Rate	93.22%	99.54%	99.75%	99.86%	99.92%	99.98%	100%	100%	100%

Table 4: The average of the five kinds of vinegar

wavelet	245	250	255	260	265	270	275	280	285
shuita	0.493	0.3582	0.3758	0.4662	0.6036	0.7336	0.8278	0.8432	0.7798
jiajia	3.8294	4.3478	4.7018	4.6418	4.7018	4.1678	4.6024	4.119	2.8532
zhenjiang	0.2518	0.544	0.878	1.267	1.716	2.1162	2.3998	2.4702	2.2708
xiaoerhei	0.648	0.3778	0.3382	0.3934	0.4968	0.5884	0.6388	0.617	0.5178
black rice	0.0488	0.1994	0.2752	0.3256	0.3626	0.3858	0.4016	0.3926	0.3592
wavelet	290	295	300	305	310	315	320	325	330
shuita	0.669	0.5866	0.5104	0.4202	0.32	0.2094	0.106	0.0608	0.039
jiajia	1.6392	0.8504	0.4188	0.2172	0.1288	0.0758	0.0454	0.0296	0.0228
zhenjiang	1.8616	1.397	1.0034	0.7098	0.4908	0.3058	0.1506	0.0886	0.0688
xiaoerhei	0.3738	0.2894	0.2264	0.1744	0.1264	0.081	0.042	0.0264	0.0204
black rice	0.3126	0.2638	0.2172	0.1716	0.128	0.0852	0.044	0.0282	0.02

-0.0125 0.0005 0.0034 0.0030 0.0010 0.0004 0.0001)

We can see that the absolute value of the former 10 elements are bigger than that of the latter 8 elements. The latter 8 elements almost go to zero. These elements correspond to the UV absorbance values which has the wavelength of 295$\sim$ 330nm. This shows that it is the absorbency values whose UV wavelength is 245 $\sim$295nm that mainly impact the first principal component , while the impact which generated by the absorbency value with UV wavelength of 295 $\sim$ 330nm can be ignored. From table 4, the average UV curves of 5 kinds of vinegar within the wavelength range of 245 $\sim$ 295nm are of much difference, and within the wave length range of 295 $\sim$ 330nm, the difference is minimal. So in the experiment, we can reduce the range of the scan UV wavelength to 245 $\sim$ 295nm.

3.3. BP Neural Network Method [4]

BP Neural Network method use error back-propagation algorithm. The data of the given sample and with ambiguity relationship can be effective classified. We designed a single hidden layer BP neural network. Since the dimension of the sample vector is 18, so the number of input layer nodes n is 18; As we experiment with 5 kinds of vinegar, the numbers of out layer nodes m is 5 and the hidden layer nodes is$\sqrt{m+n}+a$, where $a \in [1, 10]$ is a constant.

Let the training samples of Black Rice spicy vinegar, Jiajia mature vinegar, Shuita mature vinegar, Xiaoerhei grain spicy vinegar and Zhenjiang spicy vinegar be the input vectors, and

Table 5: The Identification Accuracy of BP Neural Network

Name of vinegar	Zhenjiang spicy vinegar	Xiaoerhei grain spicy vinegar	Shuita mature vinegar	Jiajia mature vinegar	Black Rice spicy vinegar
Identification Accuracy	100%	100%	100%	100%	100%

the corresponding output vector be (1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1) respectively. In the test, we classify the test sample in the class k if the kth element of the output vector is maximum.

Using sigmoid function $y = \frac{1}{1+e^{-0.5x}}$ as the active function, for the weight and the threshold, we use gradient descent momentum learning method, let the momentum coefficient be $\eta = 0.1$ and the learning rate be $\epsilon = 0.1$.

Input the 25 training samples into the net and let training goal error be 0.01. The training results of MATLAB7.4 shows that when there is 6 hidden notes, the goal error meets our qualifications at 0.00776696 only after 16 times training.Then we input the 10 test samples of given class into the net, compute the number of the samples that are classified correctly ,we can obtain the accuracy of the classification, see Table 5.

Conclusion

We studied the small sample sets of 5 kinds of vinegar, by circumrotating evaporation and ultraviolet spectrum scanning, under the conditions of wavelength at 245∼330nm, dilution ratio of evaporated liquor at 1:6, evaporation temperature at 45°C, mass concentration of refernce at 45g/L. The ultraviolet spectrum curves of vinegar at different storage time are obtained, the data are processed and analyzed by the method of pattern recognition, such as Euclid(Mahalanobis) distance, linear discriminant analysis, principal component analysis, hybrid discriminant analysis and BP neural network. The experiment result shows that the accuracy of recognition was 100% when using Euclid(Mahalanobis) distance, principal component analysis, hybrid discriminant analysis ($\lambda = 0, \eta = 1$) and BP, furthermore, we can reduce the scanning wavelength range of ultraviolet spectrum into 245∼295nm. These methods can be effective ways to identify vinegar. The reason of poor recognition accuracy of LDA is that we use the small samples set.

In the future, we will try to identify vinegar by support vector machine [5], [6].

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(σ,τ)-derivations on Jordan ideals

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Abstract Let R be a 2-torsion-free prime ring and J anonzero Jordan ideal and a subring of R. For a (σ,τ)-derivation $d : R \longrightarrow R$, we prove the following results: (1) If d is a (σ,σ)-derivation which acts as a homomorphism or as an anti-homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$; (2) If F is a generalized (σ,σ)-derivation which acts as a homomorphism or as an anti- homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$; (3) If d is a (σ,τ)-derivation which acts as a homomorphism on J, then $d = 0$ on R.

Keywords Fuzzy Prime rings, derivations, (σ,τ)-derivations, generalized derivations, generalized (σ,τ)-derivations.

§1. Introduction

Throughout the present paper, R will denote an associvative ring with center $Z(R)$. We will write for all $x, y \in R, [x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product respectively.

A ring R is said to be 2-torsion-free if whenever $2a = 0$ with $a \in R$, then $a = 0$. A ring R is said to be prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. An additive mapping $d : R \longrightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp., $d(x^2) = d(x)x + xd(x)$) holds, for all $x, y \in R$. Let σ,τ are two mappings of R. An additive mapping $d : R \longrightarrow R$ is called a (σ,τ)-derivation (resp., Jordan (σ,τ)-derivation) if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ (resp., $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$) holds, for all $x, y \in R$. Clearly, every $(1,1)$-derivation (resp, Jordan $(1,1)$-derivation), where 1 is the identity mapping on R is a derivation (resp, Jordan derivation) on R. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation associated with a derivation $d : R \longrightarrow R$ if $F(xy) = F(x)y + xd(y)$ holds, for all$x, y \in R$. In view of the definition of a (σ,σ)-derivation, the notion of generalized (σ,τ)-derivation can be extended as follows: Let σ,τ are two mappings of R. An additive mapping$F : R \longrightarrow R$ is called a generalized (σ,τ)-derivation associated with d if there exists a (σ,τ)-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds, for all $x, y \in R$. Clearly, every $(1,1)$-generalized derivation, where 1 is the identity mapping on R is a generalized derivation. In [3] Bell and Kappe proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal I of R, then $d = 0$ on R. Further Yenigul and Arac [6] obtained the a above result for α-derivation in prime rings. Recently M. Ashraf [2] extended the result for (σ,τ)-derivation in prime and semiprime rings.

As for more details and fundamental results used in this paper without mention we refer to ([1], [4] and [5]).

In the present paper our objective is to extend the above results for a (σ, σ)-derivation which acts as a homomorphism or as anti-homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R, then we will generalize the above extension for a generalized (σ, σ)-derivation.

Finally, we will prove that if $d : R \longrightarrow R$ is a (σ, τ)-derivation which acts as a homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R, then either $d = 0$ on R or $J \subseteq Z(R)$.

§2. Preliminary Results

We begin with the following lemmas which are essential to prove our main results:

Lemma 2.1. [7] If R is a ring and J a nonzero Jordan ideal of R, then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$.

Lemma 2.2. [7] Let R be a prime ring and J a nonzero Jordan ideal of R. If $a \in R$ and $aJ = 0$ or $Ja = 0$, then $a = 0$.

Lemma 2.3. [7] Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R. If $aJb = 0$, then $a = 0$ or $b = 0$.

Lemma 2.4. [7] Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R. If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.

Lemma 2.5. Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal and a subring of R. Suppose that σ, τ are automorphisms of R. If R admits a (σ, τ)-derivation d such that $d(J) = 0$, then $d = 0$ or $J \subseteq Z(R)$.

Proof. We have $d(u) = 0$, for all $u \in J$. This yields that $d(u \circ r) = 0$, for all $u \in J$ and $r \in R$. Now using the fact that $d(u) = 0$, the above expression yields that

$$\tau(u)d(r) + d(r)\sigma(u) = 0, \text{ for all } u \in J \text{ and } r \in R. \tag{2.1}$$

. Replacing r by $rs, s \in R$ in (2.1) and using (2.1), to get

$$d(r)[\sigma(s), \sigma(u)] - [\tau(r), \tau(u)]d(s) = 0, \text{ for all } u \in J \text{ and } r, s \in R. \tag{2.2}$$

. Replacing s by $sv, v \in J$ in (2.2) and using (2.2), our hypotheses yields that $d(r)\sigma(s)[\sigma(v), \sigma(u)] = 0$, for all $u, v \in J$ and $r, s \in R$ and hence $\sigma^{-1}(d(r))R[v, u] = 0$, for all $u, v \in J$ and $r \in R$. The primeness of R yields that either $d(r) = 0$ or $[v, u] = 0$, for all $u, v \in J$ and $r \in R$. If $[v, u] = 0$, for all $u, v \in J$, then it follows that J is commutative. By using Lemma (2.4), we get $J \subseteq Z(R)$.

§3. (σ, τ)-derivation as a homomorphism or as an anti-homomorphism

Let R be a ring and d a derivation of R. If $d(xy) = d(x)d(y)$ (resp., $d(xy) = d(y)d(x)$) holds, for all $x, y \in R$, then we say that d acts as a homomorphism (resp., anti-homomorphism)

on R.

Bell and Kappe [3] proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal I of R, then $d = 0$ on R.

Further, this result was extended by M. Ashraf for (σ, τ)-derivation in [2] as follows:

Theorem 3.1. [2] Let R be a prime ring, I is a nonzero right ideal of R. Suppose that σ, τ are automorphisms of R and $d : R \longrightarrow R$ is a (σ, τ)-derivation of R.

(i) If d acts as a homomorphism on I, then $d = 0$ on R.

(ii) If d acts as an anti homomorphism on I, then $d = 0$ on R.

The purpose of this section is to extend the above study to a (σ, σ)-derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R, and then we will generalize the above extension for a generalized (σ, σ)-derivation. Finally we will extend the above result for a (σ, τ)-derivation which acts as a homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R.

Theorem 3.2. Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R. Suppose that σ is an automomorphism of R, and $d : R \longrightarrow R$ is a (σ, σ)-derivation of R.

(i) If d acts as a homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$.

(ii) If d acts as an anti-homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$.

proof. Suppose that $J \nsubseteq Z(R)$.

(i) If d acts as a homomorphism on J, then we have

$$d(uv) = d(u)\sigma(v) + \sigma(u)d(v) = d(u)d(v), \text{ for all] } u, v \in J. \tag{3.1}$$

Replacing v by $vw, w \in J$ in (3.1), we get

$$d(u)\sigma(v)\sigma(w) + \sigma(u)(d(v)\sigma(w) + \sigma(v)d(w)) = d(u)(d(v)\sigma(w) + \sigma(v)d(w)).$$

Using (3.1), the above relation yields that $(d(u)-\sigma(u))\sigma(v)d(w) = 0$, for all $u, v, w \in J$, ie, $\sigma^{-1}(d(u) - \sigma(u))v\sigma^{-1}(d(w)) = 0$, for all $u, v, w \in J$, hence, $\sigma^{-1}(d(u) - \sigma(u))J\sigma^{-1}(d(w)) = 0$, for all $u, w \in J$. By using Lemma (2.3), we get either $d(u) - \sigma(u) = 0$ or $d(w) = 0$, for all $u, w \in J$. If $d(u) - \sigma(u) = 0$, for all $u \in J$, then the relation (3.1) implies that $\sigma(u)d(v) = 0$, for all $u, v \in J$. Now replace u by uw, to get $\sigma(u)\sigma(w)d(v) = 0$, for all $u, v, w \in J$, that is, $uw\sigma^{-1}(d(v)) = 0$, for all $u, v, w \in J$, and hence, $uJ\sigma^{-1}(d(v)) = 0$, for all $u, v \in J$. Thus, by Lemma (2.3), we get either $u = 0$ or $d(v) = 0$, for all $u, v \in J$. But since J is a nonzero Jordan ideal of R, we find that $d(v) = 0$, for all $v \in J$ and hence by Lemma (2.6), we get the required result.

(ii) If d acts as an anti-homomorphism on J, then we have

$$d(uv) = d(u)\sigma(v) + \sigma(u)d(v) = d(v)d(u), \text{ for all } u, v \in J. \tag{3.2}$$

Replacing u by uv in (3.2), we get

$$(d(u)\sigma(v) + \sigma(u)d(v))\sigma(v) + \sigma(u)\sigma(v)d(v) = d(v)(d(u)\sigma(v) + \sigma(u)d(v)).$$

Using (3.2), the above relation yields that

$$\sigma(u)\sigma(v)d(v) = d(v)\sigma(u)d(v), \text{ for all } u, v \in J. \tag{3.3}$$

Again replace u by $wu, w \in J$ in (3.3), we get

$$\sigma(w)\sigma(u)\sigma(v)d(v) = d(v)\sigma(w)\sigma(u)d(v), \text{ for all } u, v, w \in J. \tag{3.4}$$

In view of (3.3), the relation (3.4) yields that $[d(v), \sigma(w)]\sigma(u)d(v) = 0$, for all $u, v, w \in J$, that is, $\sigma^{-1}([d(v), \sigma(w)])u\sigma^{-1}(d(v)) = 0$, for all $u, v, w \in J$. Hence, $\sigma^{-1}([d(v), \sigma(w)])J\sigma^{-1}(d(v)) = 0$, for all $v, w \in J$.

By using Lemma (2.3), we get either $[d(v), \sigma(w)] = 0$, or $d(v) = 0$, for all $v, w \in J$. If $[d(v), \sigma(w)] = 0$, for all $v, w \in J$, then replacing v by vw in the above relation, we get

$$\sigma(v)[d(w), \sigma(w)] + [\sigma(v), \sigma(w)]d(w) = 0, \text{ for all } v, w \in J. \tag{3.5}$$

Replacing v by $v_1 v$, $v_1 \in J$ in (3.5), and using (3.5), to get $[\sigma(v_1), \sigma(w)]\sigma(v)d(w) = 0$, for all $v, v_1, w \in J$ and hence $[v_1, w]J\sigma^{-1}(d(w)) = 0$, for all $v_1, w \in J$. By Lemma (2.3), we get either $[v_1, w] = 0$ or $d(w) = 0$, for all $v_1, w \in J$.

Now let $J_1 = w \in J|[v_1, w] = 0$, for all $v_1 \in J$ and $J_2 = w \in J|d(w) = 0$. Clearly J_1 and J_2 are additive proper subgroups of J whose union is J.

Hence, by Brauer's trick, either $J = J_1$ or $J = J_2$. If $J = J_1$, then $[v_1, w] = 0$, for all $v_1, w \in J$, that is, J is commutative, and hence by Lemma (2.4), $J \subseteq Z(R)$, a contradiction. On the other hand if $J = J_2$, then by Lemma (2.6), we get the required result.

We generalize the above theorem as follows:

Theorem 3.3. Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R. Suppose that σ is an automomorphism of R and $F : R \longrightarrow R$ is a generalized (σ, σ)-derivation associated with a derivation d.

(i) If F acts as a homomorphism on J, then either $d = 0$ on R or$J \subseteq Z(R)$.

(ii) If F acts as an anti-homomorphism on J, then either$d = 0$ on R or $J \subseteq Z(R)$.

Proof. Suppose that$J \nsubseteq Z(R)$.

(i) If F acts as a homomorphism on J, then we have

$$F(uv) = F(u)\sigma(v) + \sigma(v)d(v) = F(uF(v), \text{ for all } u, v \in J. \tag{3.6}$$

Replacing v by vw, $w \in J$ in (3.6), we get

$$F(u)\sigma(v)\sigma(w) + \sigma(u)(d(v)\sigma(w) + \sigma(v)d(w)) = F(u)(F(v)\sigma(w) + \sigma(v)d(w)).$$

Using (3.6), the above relation yields that $(F(u) - \sigma(u))\sigma(v)d(w) = 0$, for all $u, v, w \in J$. Hence $\sigma^{-1}(F(u) - \sigma(w))J\sigma^{-1}(d(w)) = 0$, for all $u, w \in J$. Hence by Lemma (2.3), we get either $F(u) - \sigma(u) = 0$ or $d(w) = 0$, for all $u, w \in J$. If $F(u) - \sigma(u) = 0$, for all $u \in J$, then the relation (3.6) implies that $\sigma(u)d(v) = 0$, for all $u, v \in J$. Now, replace u by uw, to get $\sigma(u)\sigma(w)d(v) = 0$, for all $u, v, w \in J$. This implies that $uw\sigma^{-1}(d(v)) = 0$ and hence $uJ\sigma^{-1}(d(v)) = 0$, for all $u, v \in J$. thus by Lemma (2.3), we get either $u = 0$ or $d(v) = 0$. Since J is a nonzero Jordan ideal of R, we find that $d(v) = 0$, for all $v \in J$ and hence by Lemma (2.6), we get the required result.

(ii) If F acts as an anti-homomorphism on J, then we have

$$F(uv) = F(u)\sigma(v) + \sigma(u)d(v) = F(v)F(u), \text{ for all } u, v \in J. \tag{3.7}$$

Replacing u by uv in (3.7), we get

$$(F(u)\sigma(v)+\sigma(u)d(v))\sigma(v)+\sigma(u)\sigma(v)d(v)=F(v)(F(u)F(v)+\sigma(u)d(v)).$$

Using (3.7), the above relation yields that

$$\sigma(u)\sigma(v)d(v)=F(v)\sigma(u)d(u), \text{ for all } u,v\in J. \tag{3.8}$$

Again replace u by wu, $w\in J$ in (3.8), to obtain

$$\sigma(w)\sigma(u)\sigma(v)d(v)=F(v)\sigma(w)\sigma(u)d(v), \text{ for all } u,v,w\in J. \tag{3.9}$$

In view of (3.8), the relation (3.9) yields that

$$[F(v),\sigma(w)]\sigma(u)d(v)=0, \text{ for all } u,v,w\in J.$$

This implies that $\sigma^{-1}([F(v),\sigma(w)])u\sigma^{-1}(d(v))=0$, for all $u,v,w\in J$. Hence

$$\sigma^{-1}([F(v),\sigma(w)])J\sigma^{-1}(d(v))=0, \text{ for all } v,w\in J.$$

By Lemma (2.3), we get either $[F(v),\sigma(w)]=0$ or $d(v)=0$, for all $v,w\in J$.
If $[F(v),\sigma(w)]=0$, for all $v,w\in J$, then replacing v by vw in the above relation, we get

$$\sigma(v)[d(w),\sigma(w)]+[\sigma(v),\sigma(w)]d(w)=0, \text{ for all } v,w\in J. \tag{3.10}$$

Now, replacing v by v_1v, v_1 in (3.10) and using (3.10), to obtain $[\sigma(v_1),\sigma(w)]\sigma(v)d(w)=0$, for all $u,v,v_1\in J$, hence we get $[v_1,w]J\sigma^{-1}(d(w))=0$, for all $v_1,w\in J$. By using Lemma (2.3), we get either $[v_1,w]=0$ or $d(w)=0$, for all $v_1,w\in J$. Now let $J_1=\{w\in J|[v_1,w]=0,\text{ for all } v_1\in J\}$ and $J_2=\{w\in J|d(w)=0\}$. Clearly J_1 and J_1 are additive proper subgroups of J whose union is J. Hence by Brauer's trick, either $J=J_1$ or $J=J_2$.

If $J=J_1$, then $[v_1,w]=0$, for all $v_1,w\in J$, if follows that J is commutative, hence by Lemma (2.4), we get $J\subseteq Z(R)$, a contradiction. On the other hand, if $J=J_2$, then by Lemma (2.6) we get the required result.

In the next theorem we will extend Theorem (3.2) to a (σ,τ)-derivation d of a 2-torsion-free prime ring R which acts as a homomorphism on a Jordan ideal and a subring J of R.

Theorem 3.4. Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R. Suppose that σ,τ are automorphisms of R and $d:R\longrightarrow R$ is a (σ,τ)-derivation of R. If d acts as a homomorphism on J, then $d=0$ on R.

Proof. Since d acts as a homomorphism on J, then we have

$$d(uv)=d(u)\sigma(v)+\tau(u)d(v)=d(u)d(v), \text{ for all } u,v\in J. \tag{3.11}$$

Replacing v by vw, $w\in J$ in (3.11), we get

$$d(u)\sigma(v)\sigma(w)+\tau(u)(d(v)\sigma(w)+\tau(v)d(w))=d(u)(d(v)\sigma(w)+\tau(v)d(w)).$$

Using (3.11), the above relation yields that $(d(u)-\tau(u))\tau(v)d(w)=0$, for all $u,v,w\in J$, this implies that $\tau^{-1}(d(u)-\tau(u))J\tau^{-1}(d(w))=0$, for all $u,w\in J$. By using Lemma (2.3), we

get either $d(u)-\tau(u) = 0$ or $d(w) = 0$, for all $u, w \in J$. If $d(u)-\tau(u) = 0$, for all $u \in J$, we obtain $d(u) = \tau(u)$, for all $u \in J$. Then the relation (3.11) implies that $d(u)\sigma(v)+d(u)d(v) = d(u)d(v)$, for all $u, v \in J$, this yields that $d(u)\sigma(v) = 0$, for all $u, v \in J$. Now, replacing v by vw, $w \in J$, we get $d(u)\sigma(v)\sigma(w) = 0$, for all $u, v, w \in J$, that is, $\sigma^{-1}(d(u))vw = 0$, for all $u, v, w \in J$ and hence $\sigma^{-1}(d(u))Jw = 0$, for all $u, w \in J$. Hence by using Lemma (2.3), we get either $d(u) = 0$ or $w = 0$, for all $u, w \in J$. Since J is a nonzero Jordan ideal of R, we have $d(u) = 0$, for all $u \in J$, then by Lemma (2.6), we get $d = 0$ on R.

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Some another remarks on the generalization of Bernoulli and Euler numbers

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Abstract In this paper we investigate generalized Bernoulli and Euler polynomials. In section 1, we enter multiplication theorem for generalized Bernoulli and Euler numbers. Aslo in section 2, we introduce a matrix representation of B_n^{-1} and E_n^{-1}. Furthermore in section 3 we consider Euler Maclaurin summation formula for $\alpha-$ Bernoulli numbers and in section 4 we give a relation between associated stirling numbers and $B_n(a,b)$. Also in end section of this paper we consider a new method for representation of Apostol Bernoulli numbers.

Keywords Bernoulli numbers, Euler polynomials, Stirling numbers of the second kind, generating functions.

Short review of classical approaches

Bernoulli numbers were first introduced by Jacques Bernoulli (1654-1705), in the second part of his treatise published in 1713, Ars conjectandi , at the time, Bernoulli numbers were used for writing the infinite series expansions of hyperbolic and trigonometric functions. Van den berg was the first to discuss finding recurrence formulae for the Bernoulli numbers with arbitrary sized gaps (1881) [9]. Ramanujan showed how gaps of size 7 could be found, and explicitly wrote out the recursion for gaps, of size 6 [10]. Lehmer in 1934 extended these methods to Euler numbers, Genocchi numbers, and Lucas numbers (1934) [9], and calculated the 196-th Bernoulli numbers. Bernoulli polynomials play an important role in various expansions and approximation formulas which are useful both in analytic theory of numbers and in classical and numerical analysis. These polynomials can be defined by various methods depending on the applications. In particular, six approaches to the theory of Bernoulli polynomials are known, these are associated with the names of J. Bernoulli, L. Euler, P. E. Appel, A. Hurwitz, E. Lucas and D. H. Lehmer. Also Apostol and Qiu-Ming Luo defined new generalizations of Bernoulli polynomials that we have used in this paper.

§1. Generalized Raabe multiplication theorem

For a real or complex parameter α , the higher order Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the higher order Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x as well as in α, are defined

by the following generating functions:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, (|t| < 2\pi),$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, (|t| < 2\pi).$$

That the explicit formula for $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ are

$$B_n^{(\alpha)} = \sum_{k=1}^{n} \sigma(n,k)\alpha^k,$$

where

$$\sigma(n,k) = (-1)^{n-k}\frac{n!}{k!} \sum_{\substack{v_1,v_2,\cdots,v_k \in N\\ v_1+v_2+\cdots+v_k=n}} \frac{B_{v_1}B_{v_2}\cdots B_{v_k}}{(v_1v_2\cdots v_k)\, v_1!v_2!\cdots v_k!},$$

and

$$E_n^{(\alpha)}(x) = \sum_{s=0}^{n} \binom{n}{s} x^{n-s} \sum_{k=0}^{s} \frac{(-1)^k k!}{2^k} \binom{\alpha+k+1}{k} S(s,k)$$

respectively. See [1], [2].

Moreover, the higher order Bernoulli numbers $B_n^{(\alpha)}$ and higher order Euler numbers $E_n^{(\alpha)}$ are defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}\frac{t^n}{n!},$$

and

$$\left(\frac{2}{e^t+1}\right)^{\alpha} = \sum_{n=0}^{\infty} E_n^{(\alpha)}\frac{t^n}{n!}$$

respectively.

Clearly, for all nonnegative integers n, the classical Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$ are given by $B_n(x) := B_n^{(1)}(x)$ and $E_n(x) := E_n^{(1)}(x)$.

That the classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined through the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

and

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

The explicit formulas for $B_n(x)$ and $E_n(x)$, are

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

$$E_n(x) = \frac{1}{n+1}\sum_{k=1}^{n+1}(2-2^{k+1})\begin{pmatrix} n+1 \\ k \end{pmatrix} B_k x^{n+1-k},$$

where $B_k := B_k(0)$ is the k-th Bernoulli number and $E_k := E_k(1)$ is the k-th Euler number.

The Bernoulli numbers may also be calculated from

$$B_n = \lim_{x\to 0}\frac{d^n}{dx^n}\frac{x}{e^x-1}.$$

Also the Bernoulli numbers are given by the double sum

$$B_n = \sum_{k=0}^{n}\frac{1}{k+1}\sum_{r=0}^{k}(-1)^r \begin{pmatrix} k \\ r \end{pmatrix} r^n.$$

The Bernoulli numbers satisfy the sum

$$\sum_{k=0}^{n-1}\begin{pmatrix} n \\ k \end{pmatrix} B_k = 0.$$

At first we introduce necessary definitions about this matter.

Definition 1.1.([2]) Let $a, b > 0$, $a \neq b$, the generalized Bernoulli numbers $B_n(a,b)$ are defined by

$$\frac{t}{b^t - a^t} = \sum_{n=0}^{\infty}\frac{B_n(a,b)}{n!}t^n,$$

where $|t| < \frac{2\pi}{|\ln b - \ln a|}$.

Definition 1.2. Let $a, b > 0$, $a \neq b$, we define the generalized Bernoulli polynomials as

$$\frac{te^{xt}}{b^t - a^t} = \sum_{n=0}^{\infty}\frac{B_n(x;a,b)}{n!}t^n,$$

where $|t| < \frac{2\pi}{|\ln b - \ln a|}$.

Definition 1.3. For positive numbers a, b, the generalized Euler numbers $E_k(a,b)$ are defined by

$$\frac{2}{b^{2t} + a^{2t}} = \sum_{n=0}^{\infty}\frac{E_k(a,b)}{k!}t^k.$$

Definition 1.4. For any given positive numbers a, b and $x \in R$, the generalized Euler polynomials $E_k(x;a,b)$ are defined by

$$\frac{2e^{xt}}{b^t + a^t} = \sum_{n=0}^{\infty}\frac{E_k(x;a,b)}{k!}t^k.$$

Theorem 1.1.([12]) For positive numbers a, b, we have

$$B_n(x+y;a,b) = \frac{1}{2}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\left[B_k(y;a,b) + B_k(y+1;a,b)\right]E_{n-k}(x).$$

Remark 1.1. In special case if we set $b = e, a = 1, y = 0$, then we obtain

$$B_n(x) = \sum_{k=0, k\neq 1}^{n} \binom{n}{k} B_k E_{n-k}(x),$$

which is the G-S.Cheon formula. (See [1] for detail)

The term Bernoulli polynomials was used first in 1851 by Raabe [10] in connection with the following multiplication theorem

$$\frac{1}{m}\sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) = m^{-n} B_n(mx).$$

Here we give an analogues formula for generalized Bernoulli numbers.

Theorem 1.2. Let $x, y, a, b \in C$ (Complex numbers) so we have the following identity

$$\frac{1}{m}\sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\ln a + \frac{m-k+1}{m}\ln b\right) = m^{-n} B_n(mx).$$

Proof. Let us expand the function

$$\sum_{n=0}^{\infty} \frac{B_n(x,a,b)}{n!} t^n = \frac{t}{b^t - a^t} e^{xt}, b \neq a. \tag{1.1}$$

In powers of x and t and collect the coefficients of $\frac{t^n}{n!}$ as a polynomial $\Psi_n(x, a, b)$ of degree n in x :

$$F(x, t, a, b) = \sum_{n=0}^{\infty} \Psi_n(x, a, b)\frac{t^n}{n!}.$$

Suppose

$$\Psi_n(x, a, b) = A_0^{(n)} x^n + A_1^{(n)} x^{n-1} + \cdots + A_n^{(n)}.$$

That

$$A_i^{(n)} := A_i^{(n)}(a, b).$$

If we replace x by $\frac{1}{y}$ and t by ty in (1.1) we get

$$F(\frac{1}{y}, ty, a, b) = \frac{ty}{b^{ty} - a^{ty}} e^t = \sum_{n=0}^{\infty} y^n \Psi_n(\frac{1}{y}, a, b)\frac{t^n}{n!}.$$

Letting y tend to zero we obtain

$$\frac{1}{\ln b - \ln a} e^t = \sum_{n=0}^{\infty} A_0^{(n)} \frac{t^n}{n!}.$$

Hence, $A_0^{(n)} = \frac{1}{\ln b - \ln a}$ and hence $\Psi_n(x, a, b)$ is monic.

If in (1.1) we replace x by $x + \ln a^{\frac{k}{m}} b^{\frac{m-k+1}{m}}$ and sum over k and divide the result by m we get

$$\frac{1}{m}\sum_{k=0}^{m-1}\sum_{n=0}^{\infty} \Psi_n\left(x + \ln a^{\frac{k}{m}} b^{\frac{m-k+1}{m}}, a, b\right)\frac{t^n}{n!} = \frac{1}{m}\sum_{k=0}^{m-1} F\left(x + \ln a^{\frac{k}{m}} b^{\frac{m-k+1}{m}}, t, , a, b\right) = \frac{\frac{t}{m} e^{xt}}{b^{\frac{t}{m}} - a^{\frac{t}{m}}}. \tag{2.1}$$

If, instead, we replace in (1.1) x by mx and t by $\frac{t}{m}$ we obtain

$$F(mx, \frac{t}{m}, a, b) = \frac{\frac{t}{m}e^{xt}}{b^{\frac{t}{m}} - a^{\frac{t}{m}}} = \sum_{n=0}^{\infty} \frac{1}{m^n}\psi_n(mx, a, b)\frac{t^n}{n!}. \tag{3.1}$$

Identifying coefficients of $\frac{t^n}{n!}$ in (2.1), (3.1) we conclude that $\Psi_n(x, a, b)$ satisfies the functional equation

$$\frac{1}{m}\sum_{k=0}^{m-1} \Psi_n\left(x + \frac{k}{m}\ln a + \frac{m-k+1}{m}\ln b\right) = m^{-n}\Psi_n(mx).$$

Because $\Psi_n(x, a, b)$ is monic therefore proof is complete.
Now according to a next lemma we give a representation matrix for B_n^{-1} and E_n^{-1}.

§2. Matrix representation of B_n^{-1} and E_n^{-1}

Lemma 2.1.([3]) we have

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^{-1} = \frac{1}{a_0} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! a_0^{n+1}} G_n,$$

(let $a_0 \neq 0$),
where

$$G_n = \begin{vmatrix} 2a_1 & a_0 & 0 & 0 & \cdots & 0 \\ 4a_2 & 3a_1 & 0 & 0 & \cdots & 0 \\ 6a_3 & 5a_2 & 3a_0 & 3a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ (2n-2)a_{n-1} & \vdots & \vdots & \vdots & \cdots & (n-1)a_0 \\ na_n & (n-1)a_{n-1} & \vdots & \vdots & \cdots & a_1 \end{vmatrix}.$$

Now according to previous lemma and because

$$\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n\right)^{-1} = \sum_{n=0}^{\infty} \frac{B_n^{(-1)}}{n!} x^n.$$

So if we set $a_n = \frac{B_n}{n!}$ then

$$\sum_{n=0}^{\infty} \frac{B_n^{(-1)}}{n!} x^n = 1 + \sum_{n=1}^{\infty} (-1)^n G_n \frac{x^n}{n!}.$$

Now if $G_n^* := G_n$, $n \geq 1$ and $G_0^* = 1$ so $B_n^{(-1)} = (-1)^n G_n^*$.

So

$$B_n^{(-1)} =(-1)^n \begin{vmatrix} 2B_1 & B_0 & 0 & 0 & \cdots & 0 \\ 2B_2 & 3B_1 & 2B_0 & 0 & \cdots & 0 \\ B_3 & \frac{5}{2}B_2 & 4B_1 & 3B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{(2n-2)}{(n-1)!}B_{n-1} & \vdots & \vdots & \vdots & \cdots & (n-1)B_0 \\ \frac{B_n}{(n-1)!} & \frac{B_{n-1}}{(n-2)!} & \vdots & \vdots & \cdots & B_1 \end{vmatrix}.$$

Also

$$\sum_{n=0}^{\infty} \frac{E_n^{(-1)}}{n!} x^n = 1 + \sum_{n=1}^{\infty} (-1)^n Gn \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n G_n^* \frac{x^n}{n!}.$$

where $G_0^* = 1$

$$E_n^{(-1)} =(-1)^n \begin{vmatrix} 2E_1 & E_0 & 0 & 0 & \cdots & 0 \\ 2E_2 & 3E_1 & 2E_0 & 0 & \cdots & 0 \\ E_3 & \frac{5}{2}E_2 & 4E_1 & 3E_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{(2n-2)}{(n-1)!}E_{n-1} & \vdots & \vdots & \vdots & \cdots & (n-1)E_0 \\ \frac{E_n}{(n-1)!} & \frac{E_{n-1}}{(n-2)!} & \vdots & \vdots & \cdots & E_1 \end{vmatrix}.$$

§3. Euler Maclaurin summation Formula for $B_{n,\alpha}$

Let

$$g_\alpha(z) := 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha},$$

where

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+\alpha}}{2^{2k+\alpha} k! \Gamma(\alpha+k+1)}$$

is the Bessel function of the first kind order α.

The function $\frac{J_\alpha(z)}{z^\alpha}$ is an even entire function of exponential type one, we assume that α is not a negative integer. The zeros $j_k = j_k(\alpha)$ of $\frac{J_\alpha(z)}{z^\alpha}$ may then be ordered in such a way that $j_{-k} = -j_k$ and $0 < |j_1| \le |j_2| \le \cdots$. We define a sequence of polynomials $B_{n,\sigma}(x)$ by the generating function

$$\frac{e^{(x-\frac{1}{2})z}}{g_\alpha(iz/2)} = \sum_{n=0}^{\infty} B_{n,\sigma}(x) \frac{z^n}{n!}, |z| < 2|j_1|. \tag{1.3}$$

We call the polynomials $B_{n,\sigma}(x)$ the α-Bernoulli polynomials and $B_{n,\sigma}(0) =: B_{n,\sigma}$ the α-Bernoulli numbers.

To easily we see $B_{0,\sigma}(x) = 1, B_{1,\sigma}(x) = x - \frac{1}{2}, B_{2,\sigma}(x) = \left(x - \frac{1}{2}\right)^2 - \frac{1}{8(\alpha+1)}, \cdots$.
And also to easily of (1.3) we can prove

$$B'_{n,\alpha}(x) = nB_{n-1,\sigma}(x), n = 1, 2, 3, \cdots.$$

$$B_{n,\sigma}(1-x) = (-1)^n B_{n,\sigma}(x), n = 1, 2, 3, \cdots .$$

in particular $B_{n,\sigma}(1) = (-1)^n B_{n,\sigma}$. (see [4])

In mathematics, the Euler-Maclaurin formula provides a powerful connection between integrals and sums. It can be used to approximate integrals by finite sums, or conversely to evaluate finite sums and infinite series using integrals and the machinery of calculus.

In the context of computing asymptotic expansions of sums and series, usually the most useful form of the Euler-Maclaurin formula is

$$\sum_{k=a}^{b} g(k) \approx \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(g^{(2k-1)}(b) - g^{(2k-1)}(a)\right). \text{ (see [5])}$$

Where the symbol indicates that the right-hand side is a so-called asymptotic series for the left-hand side. This means that if we take the first n terms in the sum on the right-hand side , the error in approximating the left-hand side by that sum is at most on the order of the $(n+l)st$ term.

Now we will find a same formula for generalized Bernoulli numbers $B_{n,\sigma}$.

Theorem 3.1.([6]) Let f be a real function with continuous $(2k)$th derivative. Let

$$S_k = \int_1^n f(t)dt + \frac{1}{2}(f(1) + f(n)) + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!}\left(f^{(2i-1)}(n) - f^{(2i-1)}(1)\right).$$

Then

$$\sum_{i=1}^{n} f(i) = S_k - R_k,$$

where the error term is

$$R_k = \int_1^n f^{(2k)}(t)\frac{B_{2k}(\{t\})}{(2k)!}dt$$

with $B_{2k}(t)$ the Bernoulli polynomial and $\{t\} = t - \lfloor t \rfloor$ the fractional part of t.

Now we consider Euler Maclaurin summation Formula for $B_{n,\sigma}$,

The technique employs repeated integration by formula

$$B'_{n,\alpha}(x) = nB_{n-1,\alpha}, n = 1, 2, 3, \cdots$$

to create new derivatives. We start with

$$\int_0^1 f(x)dx = \int_0^1 f(x)B_{0,\alpha}(x)dx. \tag{2.3}$$

Because $B'_{1,\alpha}(x) = B_{0,\alpha}(x) = 1$ substituting $B'_{1,\alpha}(x)$ in (2.3) and integrating by parts, we obtain

$$\begin{aligned}\int_0^1 f(x)dx &= f(1)B_{1,\alpha}(1) - f(0)B_{1,\alpha}(0) - \int_0^1 f'(x)B_{1,\alpha}(x)dx \\ &= \frac{f(1)+f(0)}{2} - \int_0^1 f'(x)B_{1,\alpha}(x)dx.\end{aligned}$$

Again we have $B_{1,\alpha}(x) = \frac{1}{2}B'_{2,\alpha}(x)$ and integrating by parts

$$\int_0^1 f(x)dx = \frac{f(1)+f(0)}{2} - \frac{1}{2!}\left[f'(1)B_{2,\alpha}(1) - f'(0)B_{2,\alpha}(0)\right] + \frac{1}{2!}\int_0^1 f^{(2)}(x)B_{2,\alpha}(x)dx.$$

Using the relation

$$B_{n,\alpha}(1) = (-1)^n B_{n,\alpha}(0) = (-1)^n B_{n,\alpha}, (n = 0, 1, 2, 3, \cdots)$$

And continuing this process , we have

$$\int_0^1 f(x)dx = \frac{f(1)+f(0)}{2} - \sum_{p=1}^{q}\frac{1}{p!}B_{p,\alpha}\left[(-1)^p f^{(p-1)}(1) - f^{(p-1)}(0)\right] + \frac{1}{q!}\int_0^1 f^{(q)}(x)B_{q,\alpha}(x)dx, \tag{3.3}$$

This is the generalization of Euler-maclaurin integration formula , it assume that the function $f(x)$ has the required derivatives. The rang of integration in (2.2) my be shifted [0,1] to [1,2] by replacing $f(x)$ by $f(x+1)$. Adding such results up to $[n-1, n]$,

$$\begin{aligned}\int_0^n f(x)dx &= \frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n)\\ &- \sum_{p=1}^{q}\frac{1}{p!}B_{p,\alpha}\left[(-1)^p f^{(p-1)}(n) - f^{(p-1)}(0)\right] + \frac{1}{q!}\int_0^1 B_{q,\alpha}(x)f^{(q)}(x+v)dx.\end{aligned}$$

The terms

$$\frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n)$$

appear exactly as in trapezoidal integration or quadrature .

§4. Identity for Apostol Bernoulli numbers

Definition 4.1. The Apostol Bernoulli numbers $\beta_n(\lambda)$ are defined by means of the generating functions

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty}\beta_n(\lambda)\frac{t^n}{n!}, |t + \log\lambda| < 2\pi.$$

That $\beta_n(\lambda)$ is called Apostol Bernoulli numbers.

Lemma 4.1.([6]) Suppose that $|x| < 1$ so we have

$$\sum_{k=0}^{\infty} f(k)x^k = -\sum_{m=0}^{\infty}\frac{f^{(m-1)}(0)}{m!}\beta_m(x), |x| < 1.$$

Now according to pervious lemma we give one identity for Apostol Bernoulli numbers.

Corollary 4.1. Suppose that $|x| < 1$ so we have

$$1)\sum_{k=0}^{\infty}\cos kx^k = \sum_{n=1}^{\infty}\frac{(-1)^n\beta_{2n-1}(x)}{(2n-1)!} = \frac{1 - x\cos 1}{1 - 2x\cos 1 + x^2}, x \neq 0, |x| < 1;$$

$$2)\sum_{k=0}^{\infty}\sin kx^k=\sum_{n=1}^{\infty}\frac{(-1)^n\beta_{2n}(x)}{(2n)!}=\frac{x\sin 1}{1-2x\cos 1+x^2}, x\neq 0, |x|<1.$$

Proof. If in lemma (2.1) we set $f(x)=e^{ix}$ [where $i^2=-1$] we get

$$\sum_{k=0}^{\infty}e^{ik}x^k=-\sum_{m=1}^{\infty}\frac{i^{m-1}}{m!}\beta_m(x), |x|<1.$$

So because $e^{ik}=\cos k+i\sin k$ we get

$$\sum_{k=0}^{\infty}\cos kx^k+i\sum_{k=0}^{\infty}\sin kx^k=\sum_{n=1}^{\infty}\frac{(-1)^n\beta_{2n-1}(x)}{(2n-1)!}+i\sum_{n=1}^{\infty}\frac{(-1)^n\beta_{2n}(x)}{(2n)!}, |x|<1.$$

Also we have

$$\sum_{n=0}^{\infty}r^n\cos n\theta=\frac{1-r\cos\theta}{1-2r\cos\theta+r^2},$$

and

$$\sum_{n=0}^{\infty}r^n\sin n\theta=\frac{1-r\sin\theta}{1-2r\cos\theta+r^2}.$$

So

$$\sum_{n=1}^{\infty}\frac{(-1)^n\beta_{2n-1}(x)}{(2n-1)!}=\frac{1-x\cos 1}{1-2x\cos 1+x^2},$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n\beta_{2n}(x)}{(2n)!}=\frac{x\sin 1}{1-2x\cos 1+x^2}.$$

Therefore proof is complete.

§5. Asymptotic relation between 2-associated stirling number and $B_n(a,b)$

We defined the generalized 2-associated stirling numbers by

$$\sum_{n=k}^{\infty}S_2^*(n,a,b,k)\frac{t^n}{n!}=\frac{b^t-(1+t)a^t}{a^{kt}k!}, a\neq 0,$$

where k and r are positive integers. It is clear that if we set $b=e$ and $a=1$ then

$$\sum_{n=k}^{\infty}S_2^*(n,k)\frac{t^n}{n!}=\frac{(e^t-1-t)^k}{k!},\ \text{see [7]}.$$

We give asymptotic expansion of certain sums for generalized 2-associated stirling numbers of the second kind, Bernoulli numbers, Euler numbers by Darboux's method.

Lemma 5.1. Assume that

$$f(t)=\sum_{n=0}^{\infty}a_nt^n$$

is an analytic function for $|t| < r$ and with a finite number of algebraic singularities on the circle $|t| = r$, $\alpha_1, \alpha_2, \cdots, \alpha_l$ are singularities of order ω is the highest order of all singularities. Then

$$a_n = \left(\frac{n^{\omega-1}}{\Gamma(\omega)}\right)\left(\sum_{k=1}^{l} g_k(\alpha_k)\alpha_k^{-n} + O(r^{-n})\right), \text{ see [8].}$$

Where $\Gamma(w)$ is the gamma function, and

$$g_k(\alpha_k) = \lim_{t\to\alpha_k}\left(1 - \left(\frac{t}{\alpha_k}\right)\right)^{\omega} f(t).$$

Theorem 5.1. Suppose that $n \geq 1$ and $k \geq 1$, where k is fixed, when $n \to \infty$, we have (here let $\ln \frac{b}{a}$ is a algebraic number)

$$\sum_{p+q=2n} \frac{S_2^*(p+k,a,b,k)B_q(a,b)}{(p+k)!q!} \approx \frac{2(-1)^{n+k+1}}{\left(\frac{2\pi}{|\ln b - \ln a|}\right)^{2n} k!}.$$

Proof. It is clear that according to definition we have

$$\sum_{p=0}^{\infty} S_2^*(p+k,a,b,k)\frac{t^p}{(p+k)!}\sum_{q=0}^{\infty} B_q(a,b)\frac{t^q}{q!} = \frac{\left(b^t-(1+t)a^t\right)^k}{k!t^{k-1}(b^t-a^t)a^{kt}}, a \neq b.$$

Let

$$f(t) = \frac{\left(b^t-(1+t)a^t\right)^k}{k!t^{k-1}(b^t-a^t)a^{kt}},$$

then $f(t)$ is analytic for $|t| < \frac{2\pi}{|\ln b - \ln a|}$ and with two algebraic singularities on the circle $|t| = \frac{2\pi}{|\ln b - \ln a|}$.

$\alpha_1 = \frac{2\pi i}{|\ln b - \ln a|}$ and $\alpha_2 = \frac{-2\pi i}{|\ln b - \ln a|}$ are singularities of order 1. To easily we can compute

$$\lim_{t\to\frac{2\pi i}{|\ln b-\ln a|}}\left(1 - \frac{t}{\frac{2\pi i}{|\ln b - \ln a|}}\right) f(t) = \lim_{t\to\frac{-2\pi i}{|\ln b-\ln a|}}\left(1 + \frac{t}{2\pi i}\right) f(t) = \frac{(-1)^{k+1}}{k!|\ln b - \ln a|}.$$

It follows from [8] that

$$\begin{aligned}\sum_{p+q=2n} \frac{S_2^*(p+k,a,b,k)B_q(a,b)}{(p+k)!q!} &= \frac{1}{\Gamma(1)}\left\{\frac{(-1)^{k+1}}{k!}\left[\left(\frac{2\pi!}{|\ln b - \ln a|}\right)^{-n}\right.\right.\\ &\quad \left.\left. + \left(\frac{-2\pi!}{|\ln b - \ln a|}\right)^{-n}\right] + O\left(\left(\frac{-2\pi!}{|\ln b - \ln a|}\right)^{-n}\right)\right\}.\end{aligned}$$

So we have

$$\sum_{p+q=2n} \frac{S_2^*(p+k,a,b,k)B_q(a,b)}{(p+k)!q!} \approx \frac{(-1)^{k+1}\left[(i)^{2n}+(-i)^{2n}\right]}{\left(\frac{2\pi}{|\ln b-\ln a|}\right)^{2n} k!} = \frac{2(-1)^{n+k+1}}{\left(\frac{2\pi}{|\ln b-\ln a|}\right)^{2n} k!}.$$

Therefore proof is complete.

§6. New method for representation of Apostol Bernoulli and Euler polynomials

Let $\sigma(n)$ denote the set of partitions of n (a nonnegative integer) usually denoted by $1^{k_1}2^{k_2}3^{k_3}\cdots n^{k_n}$ with $\sum ik_i = n$

For nonnegative integral vector $\overline{k} = (k_1, k_2, \ldots, k_n)$, the multinomial coefficient $\binom{x}{k}$ as usual, is defined by

$$\binom{x}{k} = \frac{(x)_{|\overline{k}|}}{\prod(k_i)!},$$

where the finite product $\prod$ runs over i from 1 to n, $(x)_k$ stands for the all factorial notation, and $|\overline{k}|$ represents the coordinate sum for the vector $\overline{k} = (k_1, k_2, \ldots, k_n)$. Now let

$$[g(t)]^x = \sum_{n\geq 0} A_n(x)t^n,$$

where x is an arbitrary complex number independent of t.

Theorem 6.1.([11]) For arbitrary complex number x and y,

$$A_n(xy) = \sum_{\sigma(n)} \binom{x}{k} \prod [A_i(y)]^{k_i}$$

So according to these theorems we have the following results.

Corollary 6.1. Let α, β are complex numbers then we have the following identities

$$B_n^{(\alpha\beta)} = n! \sum_{\sigma(n)} \binom{\alpha}{k} \prod \left(\frac{B_i^{(\beta)}}{i!}\right)^{k_i},$$

$$E_n^{(\alpha\beta)} = n! \sum_{\sigma(n)} \binom{\alpha}{k} \prod \left(\frac{E_i^{(\beta)}}{i!}\right)^{k_i}.$$

Proof. Let $A_n(\alpha) = \frac{B_n^{(\alpha)}}{n!}$ in theorem 6.1.

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